Chapter 28

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Game Theory

Introduction. In this introduction we offer two examples of two-person games. The first game has a dominant strategy equilibrium. The second game is a zero-sum game that has a Nash equilibrium in pure strategies that is not a dominant strategy equilibrium.

Example: Albert and Victoria are roommates. Each of them prefers a clean room to a dirty room, but neither likes housecleaning. If both clean the room, they each get a payoff of 5. If one cleans and the other doesn't clean, the person who does the cleaning has a utility of 2, and the person who doesn't clean has a utility of 6. If neither cleans, the room stays a mess and each has a utility of 3. The payoffs from the strategies "Clean" and "Don't Clean" are shown in the box below.

Clean Room–Dirty Room

		Victoria: Clean	Victoria: Don't Clean
Albert	Clean	5, 5	2, 6
	Don't Clean	6, 2	3, 3

In this game, whether or not Victoria chooses to clean, Albert will get a higher payoff if he doesn't clean than if he does clean. Therefore "Don't Clean" is a dominant strategy for Albert. Similar reasoning shows that no matter what Albert chooses to do, Victoria is better off if she chooses "Don't Clean." Therefore the outcome where both roommates choose "Don't Clean" is a dominant strategy equilibrium. This is true despite the fact that both persons would be better off if they both chose to clean the room.

Example: This game is set in the South Pacific in 1943. Admiral Imamura must transport Japanese troops from the port of Rabaul in New Britain, across the Bismarck Sea to New Guinea. The Japanese fleet could either travel north of New Britain, where it is likely to be foggy, or south of New Britain, where the weather is likely to be clear. U.S. Admiral Kenney hopes to bomb the troop ships. Kenney has to choose whether to concentrate his reconnaissance aircraft on the Northern or the Southern route. Once he finds the convoy, he can bomb it until its arrival in New Guinea. Kenney's staff has estimated the number of days of bombing time for each of the outcomes. The payoffs to Kenney and Imamura from each outcome are shown in the box below. The game is modeled as a "zero-sum game:" for each outcome, Imamura's payoff is the negative of Kenney's payoff.

The Battle of the Bismarck Sea

		Imamura	
		North	South
Kenney	North	2, −2	2, −2
	South	1, −1	3, −3

This game does not have a dominant strategy equilibrium, since there is no dominant strategy for Kenney. His best choice depends on what Imamura does. The only Nash equilibrium for this game is where Imamura chooses the northern route and Kenney concentrates his search on the northern route. To check this, notice that if Imamura goes North, then Kenney gets an expected two days of bombing if he (Kenney) chooses North and only one day if he (Kenney) chooses South. Furthermore, if Kenney concentrates on the north, Imamura is indifferent between going north or south, since he can be expected to be bombed for two days either way. Therefore if both choose "North," then neither has an incentive to act differently. You can verify that for any other combination of choices, one admiral or the other would want to change. As things actually worked out, Imamura chose the Northern route and Kenney concentrated his search on the North. After about a day's search the Americans found the Japanese fleet and inflicted heavy damage on it.*

Chapter 29

NAME________________________

Game Applications

Introduction. As we have seen, some games do not have a Nash equilibrium in pure strategies. But if we allow for the possibility of Nash equilibrium in mixed strategies, virtually every game of the sort we are interested in will have a Nash equilibrium.

The key to solving for such equilibria is to observe that if a player is indifferent between two strategies, then he is also willing to choose randomly between them. This observation will generally give us an equation that determines the equilibrium.

Example: In the game of baseball, a pitcher throws a ball toward a batter who tries to hit it. In our simplified version of the game, the pitcher can pitch high or pitch low, and the batter can swing high or swing low. The ball moves so fast that the batter has to commit to swinging high or swinging low before the ball is released.

Let us suppose that if the pitcher throws high and the batter swings low, or the pitcher throws low and the batter swings high, the batter misses the ball, so the pitcher wins. Also suppose that if the pitcher throws high and the batter swings high, the batter always connects, but if the pitcher throws low and the batter swings low, the batter will connect only half the time.

This scenario leads us to the following payoff matrix, where if the batter hits the ball he gets a payoff of 1 and the pitcher gets 0, and if the batter misses, the pitcher gets a payoff of 1 and the batter gets 0.

Simplified Baseball

		Batter	
		Swing Low	Swing High
Pitcher	Pitch High	1, 0	0, 1
	Pitch Low	.5, .5	1, 0

This game has no Nash equilibrium in pure strategies. There is no combination of actions taken with certainty such that each is making the best response to the other's action. The batter always wants to swing the same place the pitcher throws, and the pitcher always wants to throw to the opposite place. What we can find is a pair of equilibrium mixed strategies.

In a mixed strategy equilibrium each player's strategy is chosen at random. The batter will be willing to choose a random strategy only if the expected payoff to swinging high is the same as the expected payoff to swinging low.

The payoffs from swinging high or swinging low depend on what the pitcher does. Let π_P be the probability that the pitcher throws high and $1 - \pi_P$ be the probability that he throws low. The batter realizes that if he swings high, he will get a payoff of 0 if the pitcher throws low and 1 if the pitcher throws high. The expected payoff to the batter is therefore π_P.

If the pitcher throws low, then the only way the batter can score is if the pitcher pitches low, which happens with probability $1 - \pi_P$. Even then the batter only connects half the time. So the expected payoff to the batter from swinging low is $.5(1 - \pi_P)$.

These two expected payoffs are equalized when $\pi_P = .5(1 - \pi_P)$. If we solve this equation, we find $\pi_P = 1/3$. This has to be the probability that the pitcher throws high in a mixed strategy equilibrium.

Now let us find the probability that the batter swings low in a mixed strategy equilibrium. In equilibrium, the batter's probability π_B from swinging low must be such that the pitcher gets the same expected payoff from throwing high as from throwing low. The expected payoff to the pitcher is the probability that the batter does *not* score.

If the pitcher throws high, then the batter will not connect if he swings low, but will connect if he swings high, so the expected payoff to the pitcher from pitching high is π_B.

If the pitcher throws low, then with probability $(1 - \pi_B)$, the batter will swing swing high, in which case the pitcher gets a payoff of 1. But when the pitcher throws low, the batter will swing low with probability π_B and connect half the time, giving a payoff to the pitcher of $.5\pi_B$. Therefore the expected payoff to the pitcher from throwing low is $(1 - \pi_B) + .5\pi_B = 1 - .5\pi_B$. Equalizing the payoff to the pitcher from throwing high and throwing low requires $\pi_B = 1 - .5\pi_B$. Solving this equation we find that in the equilibrium mixed strategy, $\pi_B = 2/3$.

Summing up, the pitcher should throw low two-thirds of the time, and the batter should swing low two-thirds of the time.