# Economics 661: Theoretical IO
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Updated: May 11, 2006

# 1 Lecture 1: January 31, 2006

## 1.1 Product Selection, Quality and Advertising

- Does a monopolist produce the right amount of variety? How do we model product differentiation? What about issues of incomplete information?
- We'll start by letting product attributes (location, availability, quality, etc) as opposed to the products themselves, enter the utility function.

**Vertical Product Differentiation**

- The main assumptions of VPD is that consumers agree on which good is higher quality, but have different willingness to pay.
- Quality is indexed by $s \in \Re_{++}$. Good are substitutes so in the 2 good case, the consumers won't want to buy both goods.
- Preferences are defined by,

  $$u = \theta s - p,$$

  if a consumer of type $\theta$ buys a good of quality $s$ at price $p$. $u = 0$ if they do not buy. $\theta \sim F(\cdot)$. Suppose there are $N$ consumers.
- 1 good case. Buy iff:

  $$\theta s - p \geq 0 \Longleftrightarrow \theta \geq \frac{p}{s}.$$

  This induces demand:

  $$D(p) = N * Pr\{\theta \geq \frac{p}{s}\} = N[1 - F(\frac{p}{s})].$$

  Note we can drop the $N$ WLOG.
- Now consider the 2 good case. A monopolist offers two goods defined by the quality/price pairs $(s_1, p_1)$, $(s_2, p_2)$.
- A consumer of type $\theta$ buys good 1 only if:
  - (c1) $\theta s_1 - p_1 \geq 0 \Longleftrightarrow \theta \geq \frac{p_1}{s_1}$.
  - (c2) $\theta s_1 - p_1 \geq \theta s_2 - p_2 \Longleftrightarrow \theta \leq \frac{p_2 - p_1}{s_2 - s_1}$.
- A consumer of type $\theta$ buys good 2 only if:
  - (c3) $\theta s_2 - p_2 \geq 0$.
  - (c4) $\theta s_2 - p_2 \geq \theta s_1 - p_1$.

- So we assume $s_2 > s_1$, good 2 is the higher quality good. Clearly if $p_1 > p_2$, then everyone would buy good 2 since it is cheaper and better. Also, combining (c1) and (c2):

$$\frac{p_1}{s_1} \leq \theta \leq \frac{p_2 - p_1}{s_2 - s_1}.$$

$$\frac{p_2 - p_1}{s_2 - s_1} \geq \frac{p_1}{s_1}. \quad (*)$$

- We can see the result (*) graphically in G-1.1. Clearly, if the point E is below the axis, no one buys the low-quality good. For consumers to buy both goods (some buy 1 and some buy 2), we must see something likg G-1.2. The point C is $(p_2 - p_1)/(s_2 - s_1)$ and A is $p_1/s_1$, confirming (*).

- Demand functions in the 2 good case of VPD are thus:

$$D_1(p_1, p_2) = N[F(\frac{p_2 - p_1}{s_2 - s_1}) - F(\frac{p_1}{s_1})].$$

$$D_2(p_1, p_2) = N[1 - F(\frac{p_2 - p_1}{s_2 - s_1})].$$

**Horizontal Product Differentiation**

- The main assumptions of HPD is that consumers DO NOT agree on which good is higher quality. They have different tastes.

- The simplest model of HPD is the linear road model where firms located at either $x = 0$ or $x = 1$, the end points of main street in a small town with consumers distributed uniformly over the interval. Transportation cost is $t$, consumers value the good at $V$, and pay either $p_0$ or $p_1$. The the equation of the marginal man is thus:

$$u_0 = u_1 \Longleftrightarrow V - p_0 - t\tilde{x} = V - p_1 - t(1 - \tilde{x}).$$

$$\tilde{x} = \frac{p_1 - p_0}{2t} + \frac{1}{2}.$$

- Clearly if $p_0 = p_1$, $\tilde{x} = \frac{1}{2}$ and the firms split the consumers equally independent of $t$. If $p_1 > p_0$ and we increase $t$, then $\tilde{x} \to 0$. That is, even though good zero is cheaper, transportation costs become more important than price and consumers start to head towards good one.

- It is also possible that consumers choose to buy from neither of the two firms if prices or transportation costs are too high. See G-1.3. These critical values become:

$$x_0^* = \frac{V - p_0}{t}, \quad x_1^* = 1 - \frac{V - p_1}{t}.$$

- Demands are thus:

$$D_0(p_0, p_1) = N * [Max\{Min\{x_0^*, \tilde{x}, 1\}, 0\}].$$

$$D_1(p_0, p_1) = N * [1 - Min\{Max\{x_1^*, \tilde{x}, 0\}, 1\}].$$

**Product Selection by a Monopolist**

- Consider a model of VPD where a monopolist is choosing how high a quality to product. We will compare his quantity/quality choice to that of a social planner.
- Inverse demand: $P(q, s)$.
- Cost: $C(q, s)$.
- The Social Planner's Problem (SP):

$$Max_{q,s} \quad \underbrace{\int_0^q P(x, s)dx}_{Gross\ CS} - C(q, s).$$

FOCs:

$$(SP1): \quad P(q, s) = \frac{\partial C(q, s)}{\partial q}.$$

$$(SP2): \quad \int_0^q \frac{\partial P(x, s)}{\partial s} dx = \frac{\partial C(q, s)}{\partial s}.$$

- Monopolist's Problem (MP):

$$Max_{q,s} \ \ P(q, s)q - C(q, s).$$

FOCs:

$$(MP1): \quad P(q, s) + q\frac{\partial P(q, s)}{\partial q} = \frac{\partial C(q, s)}{\partial q}.$$

$$(MP2): \quad \frac{\partial P(q, s)}{\partial s} q = \frac{\partial C(q, s)}{\partial s}.$$

- Using (SP2) and (MP2), for a given $q$, a monopolist undersupplies (oversupplies) quality when:

$$\int_0^q \frac{\partial P}{\partial s} dx >_{(<)} q\frac{\partial P}{\partial s}.$$

Though it is possible that this holds with equality, it's an event of measure zero, so the monopolist is never producing the social planner's optimal quality.

# 2 Lecture 2: February 7, 2006

## 2.1 More on HPD

- There are two classes of models of horizontal product differentiation. One is spatial models, like hotelling's linear road. The other is called the Representative Consumer model where the aggregate demand for a group of consumer is assumed to come from one representative agent. Then solve for the utility function which would have generated that demand and take those preferences to be representative of the whole group of consumers. So we might solve:

$$(x_1^*(p), x_2^*(p), \ldots, x_n^*(p))' \in arg\ max_x\ \{u(x_1, \ldots, x_n) | \sum_{i=1}^{n} p_i x_i \leq Y\}.$$

- The problem with representative consumer models is they lack the spatial dimension on an individual basis. Spatial models however also have issues.

## 2.2 Optimal Time of Development

- In this section, we would like to see if a monopolist develops a product at the optimal time, at least compared to what a social planner would do.
- A SP would introduce a new product if the total social surplus from doing so exceeded the total cost of production.
- However it may be that the monopolist's profits are less than this surplus. Indeed it might be that:

$$W > f > \pi^m,$$

  or the social welfare exceeds the fixed cost of development (so the SP wants to develop), but that fixed cost exceeds the monopolist's profits (so the Monopolist would not develop).
- To illustrate this point, we consider a simple linear city model with a single monopolist and consumers distributed uniformly over the interval $[0, 1]$. Assume $t = 1$, preferences are $u = V - x - p$, and $V$ is high enough so that the firm always wants to sell to everyone.
- The monopolist is located initially at zero and as in G-2.1, since he sells to everyone, the highest price he can set is $p_0^m$. Raising the price any higher would cut those living at one out of the market. The line $s - x$ with slope $-1$ is plotted in the graph.
- The total transportation cost for the consumers initially is the shaded region, which has an area of $\frac{1}{2}$.
- Now suppose the monopolist considers opening a store at one. The graph G-2.2 now includes $s - x$ and $s - (1 - x)$. Those to the right of $\frac{1}{2}$ buy from the monopolist at one

and to the left buy from zero. The transportation cost is now smaller and is shown in the graph shaded with circles. The area of this region is $\frac{1}{4}$.

- So the gains from the SP's point of view of the new store at one is $\frac{1}{4}$. Thus, if the costs of opening the store are one are less than $\frac{1}{4}$, the SP would open it.
- HOWEVER, the monopolist has a different point of view. By opening the new store and still selling to all consumers, he can raise is price to $p_1^m$. See G-2.3. This price is equal to:

$$p_1^m = p_0^m + \frac{1}{2}.$$

  Thus, profits for the monopolist rise by $\frac{1}{2}$. So the monopolist will develop, ie, open the store at one, if the fixed costs are less than $\frac{1}{2}$.
- Comparing the SP to the monopolist, the monopolist opens the store at one more often than the SP would want to! So the monopolist *oversupplies.* For,

$$f \in (\frac{1}{4}, \frac{1}{2}),$$

  the monopolist develops while the SP would not.
- **Punchline** The monopolist creates a larger variety of products than is socially optimal. The reason is that by introducing substitute products, the constraint the monopolist faces of selling to all consumers is weakened.

## 2.3 Quality and Information

- There are two main types of goods:
  - (1) Experience Goods: you have to buy these once and actually use them to know their quality.
  - (2) Search Goods: just by seeing it the store, you know it's quality without actual purchase.

Note that if you can contract on quality via warantees or guarantees, that would be classified as a search good. We'll deal with experience goods in what follows.

### Akerlof - Market for Lemons

- Cars for sale are either high or low quality. Buyers and sellers value high and low quality cars at $(B_h, B_l)$, $(S_h, S_l)$, with:

$$B_h > B_l,\ S_h > S_l,\ B_h > S_h,\ B_l > S_l,$$

so trade should occur.

- Since buyers can not distinguish high and low quality, all they know is that $\alpha$ is the proportion of high quality cars on the lot. Thus, if both types of cars get sold, buyers require:
$$p \leq \alpha B_h + (1-\alpha)B_l.$$
Sellers require:
$$p > S_l, \; and \; p > S_h.$$

- This conditions imply:
$$S_h < \alpha B_h + (1-\alpha)B_l,$$
which is very easy to violate if, for instance, $\alpha$ is relatively small.

**Continuous Example - Insurance**

- Let $\theta \in \Theta$, denote the type of consumer, with $\theta \sim F(\cdot)$. Assume:
$$\theta = Prob\{L\},$$
where $L$ is some monetary loss. So the higher $\theta$ consumers are more prone to accidents.

- Let $V(\theta)$ denote a consumer of type $\theta$'s willingness to pay for full insurance against the loss. Note $V' > 0$.

- Let $C(\theta) = \theta L$ be the cost of supplying insurance.

- Agents are risk averse so $V(\theta) > C(\theta)$ for $\theta > 0$. Thus the social optimum involves everyone getting insurance.

- An equilibrium is a price, $p^*$, and a set, $B^* \in \Theta$ where the consumers in $B^*$ buy insurance. We need two conditions:
  - (1) $\forall\, \theta \in B^*,\; V(\theta) \geq p^* \geq \int_{\theta \in B^*} \theta L \, dF.$
  - (2) $\forall\, \theta \notin B^*,\; V(\theta) < p^*.$

  So we need those with a high willingness to pay to buy, and that price must exceed the cost of supplying that insurance. We also need that those that do not buy to have a small willingness to pay.

## 2.4 Quality Choice by a Monopolist

- What if the monopolist can choose the quality but consumers do not observe it?

- We assume this is a one shot game and the good is an experience good. For example, the monopolist can produce either a good or bad MP3 player.

- The firm chooses a price and quality combination $(p, a)$.

- Agent's utility: $u(p,q,a)$, where $q$ is the quantity of units purchased by the consumer.
- Monopolist's profits:

$$\pi(p,q,a) = pq - c(q,a),\ c_q > 0,\ c_a > 0.$$

- So in equilibrium suppose we had a triple, $(p^*, a^*, q^*)$ with $a^* > a_{min}$. This induces profits of:

$$\pi(p^*, q^*, a^*) = p^*q^* - c(q^*, a^*).$$

  But since consumers do NOT observe quality, and costs are increasing in $a$, then in a Perfect Bayesian equilibrium, the firm has an optimal deviation to:

$$\pi(p^*, q^*, a_{min}) > \pi(p^*, q^*, a^*).$$

  So only the lowest quality products would be supplied.

- So what can a monopolist do about this? All his consumers will think that he is going to supply a low quality product (and possibly not even buy from him as a result). Several solutions have been proposed including reputation (in repeated games), signalling, guarantees, etc. We'll look at one of these next.

**Milgrom and Roberts, JPE, 1986**

- Suppose a monopolist introduces a new experience good and signals the quality by engaging in costly advertising.
- Two possible quality levels: $(H, L)$. The firm picks a price, $P$ and advertising level, $A$.
- Observing $(P, A)$, the consumers decide on their initial purchases.
- Firm profits:

$$\Pi(P, q, \sigma) - A,$$

  which is the present value of profits for a product with quality $q$, price $P$, and advertising level $A$. Assume $\sigma = Prob\{q = H|A, P\}$. So if $(A, P)$ is observed, $\sigma$ is the belief that the product is of high quality.

- For simplicity, suppose only two price and advertising pairs are available to the firm: $(P, A)$ and $(P', A')$. We need some assumptions on $\sigma$ to determine our PBE. Suppose if a monopolist chooses to produce a high quality product, he sets $(P, A)$, and if he chooses a low quality product, he sets $(P', A')$. Then for a consumer, if $(P, A)$ is observed, $\sigma = 1$ and if $(P', A')$ is observed, $\sigma = 0$. If anything else is observed, we can choose $\sigma$ to be whatever we want, or make equilibrium refinements.
- Denote:

$$\pi(p, q, L) = \Pi(p, q, \sigma = 0),$$

$$\pi(p, q, H) = \Pi(p, q, \sigma = 1),$$

so these are just realized profits for two different beliefs. Also denote:

$$p_Q^q = arg\ max_p\ \{\pi(p, q, Q)\}.$$

So the superscript, $q$, is the actual quality of the good, while the subscript, $Q$, is the belief.

- **Proposition** $\exists$ a separating sequential equilibrium iff for some $(P, A) \geq 0$,

  $$\pi(P, H, H) - A \geq \pi(P_L^H, H, L),$$

  and,

  $$\pi(P, L, H) - A \leq \pi(P_L^L, L, L).$$

  So the first condition says that a monopolist that can produce at high quality will find it optimal to do so and signal with an advertising choice $A > 0$. His profits by doing this are higher than if he didn't signal with $A$ resulting in consumers believing he was a low quality producer. The second condition says that for a low quality firm, it is not profitable to signal he is high quality (fake it) by advertising at $A > 0$. Combining these inequalities:

  $$\pi(P, H, H) - \pi(P_L^H, H, L) \geq A \geq \pi(P, L, H) - \pi(P_L^L, L, L).$$

- Clearly there is a social waste if advertising is costly. Engaging in $A$ is only a signal of quality, which could be avoided if consumers could observe quality directly.

# 3 Lecture 3: February 14, 2006

## 3.1 Product Differentiation in an Oligopoly Environment

- Consider again the linear city model with consumers of type, $\theta \sim U[0,1]$. Payoffs are:

$$u(p_0, p_1|\theta) = V - t\theta - p_0,$$

if he purchases from firm zero and:

$$u(p_0, p_1|\theta) = V - t(1-\theta) - p_1,$$

if he purchases from firm one. Otherwise, with no purchases, his payoff is 0.

- Assume firms have marginal cost, $c$. Assume $V$ is high enough so that everyone buys from one of the firms in equilibrium.

- Marginal man defines:

$$V - t\theta^* - p_0 = V - t(1-\theta^*) - p_1,$$

$$\theta^* = \frac{p_1 - p_0}{2t} + \frac{1}{2}.$$

- This means that profits for firm $i$:

$$\pi^i(p_i, p_j) = (p_i - c)[\frac{p_j - p_i}{2t} + \frac{1}{2}].$$

FOC:

$$\frac{p_j - p_i}{2t} + \frac{1}{2} - \frac{p_i - c}{2t} = 0$$

Or,

$$p_i^*(p_j) = \frac{p_j + c + t}{2}.$$

- Since costs are symmetric, $p_i = p_j$, or:

$$p_i^* = p_j^* = p^* = t + c.$$

Which induces:

$$\pi^i(p^*, p^*) = \frac{p^* - c}{2} = \frac{t}{2}.$$

Note this does NOT depend on $V$!

- So $t$, the transportation cost, is a measure of product differentiation. As $t$ gets larger, the products are seen as more different from one and other. Note:

$$\frac{\partial \pi^i}{\partial t} > 0,$$

so profits are increasing in the degree of differentiation under the assumption that everyone buys the good.

- What about welfare implications? Total welfare is:

$$W = V - \underbrace{2\int_0^{1/2} tx\ dx}_{trany\ costs} - c.$$

See G-3.1. Thus,

$$W = V - 2[\frac{t}{2}x^2]_0^{1/2} - c.$$

$$W = V - \frac{t}{4} - c.$$

Consumer surplus is total welfare less firm profits:

$$CS = W - 2\pi^i = V - \frac{t}{4} - c - t = V - \frac{5}{4}t - c.$$

- What about our assumption that all consumers buy? Consider the consumer most likely NOT to buy, the one located at 1/2. He will BUY only if:

$$V - \frac{1}{2}t - (t + c) \geq 0.$$

$$V - \frac{3}{2}t - c \geq 0.$$

- If we suppose that this last condition does NOT hold, then some consumers do not buy and each firm on each end point acts as a monopolist on the consumers located near to him. For firm zero, there is a critical consumer such that:

$$\theta^* = \frac{V - p_0}{t},$$

or he is just indifferent between buying from zero and NOT BUYING AT ALL! Profits are:

$$\pi^0(p_0) = (p_0 - c)[\frac{V - p_0}{t}].$$

FOC:

$$\frac{V - p_0}{t} - \frac{p_0 - c}{t} = 0$$

$$p_0^* = \frac{V + c}{2},$$

$$\pi^0(p_0^*) = \frac{(V - c)^2}{4t}.$$

Now note that

$$\frac{\partial \pi^0}{\partial t} < 0,$$

so profits are decreasing in the degree of differentiation. This is because firms are acting as monopolists on each end of the city so they do not compete over the same customers.

## 3.2 The Circular City Model

- Consider consumers uniformly distributed around a unit circle. This has some nice features of eliminating the consumers located at the "end of the road." All consumers are completely symmetric.
- Suppose each consumer's most preferred product is $l^*$, (her location). If a consumer buys a product other than $l^*$, their utility is:

$$u(l_i, l^*) = V - t|l_i - l^*|.$$

- Assume symmetric costs (marginal cost, $c$, and fixed cost, $f$) and firms are always equidistant from each other. Asume $V$ is high enough or $t$ low enough that everyone buys.
- Assume $p_i$ is firm $i$'s price, there are $n$ firms, and the distance between firms is thus $1/n$.
- Marginal man:

$$p_i + tl^* = p + t(\frac{1}{n} - l^*).$$

So:

$$l^* = \frac{p - p_i}{2t} + \frac{1}{2n}.$$

- Demand for firm $i$ is twice $l^*$ because he gets the consumers on either side of him:

$$D_i(p_i, p) = 2l^* = \frac{p - p_i}{t} + \frac{1}{n}.$$

- Firm's problem:

$$Max_{p_i} \ \{\pi^i = (p_i - c)[\frac{p - p_i}{t} + \frac{1}{n}] - f\}.$$

FOC:

$$\frac{p - p_i}{t} + \frac{1}{n} - \frac{p_i - c}{t} = 0.$$

Set $p = p_i = p^*$ by symmetry:

$$\frac{1}{n} = \frac{p^* - c}{t}.$$

$$p^* = \frac{t}{n} + c.$$

- Firm profits:

$$\pi^i(p^*) = (p^* - c)[\frac{p^* - p^*}{t} + \frac{1}{n}] - f.$$

$$\pi^i(p^*) = \frac{t/n + c - c}{n} - f.$$

$$\pi^i(p^*) = \frac{t}{n^2} - f.$$

- Firm entry condition:

$$\pi^i(p^*) > 0 \Leftrightarrow \frac{t}{n^2} > f.$$

So in equilibrium, $\pi^i = 0$, or:

$$n = \sqrt{\frac{t}{f}}.$$

- So note that firms price ABOVE marginal cost, but still can earn zero profits in equilibrium. If $n$ is NOT a whole number, firms may earn positive profits in equilibrium.

- Social Planner's solution. The SP seeks to minimize the transportation costs and fixed costs. So:

$$Min_n \ \{nf + t[2n \int_0^{1/(2n)} xdx]\}$$

$$Min_n \ \{nf + tn\frac{1}{4n^2}\}$$

$$Min_n \ \{nf + \frac{t}{4n}\}$$

FOC:

$$f - \frac{t}{4n^2} = 0 \Rightarrow n = \sqrt{\frac{t}{4f}} = \frac{1}{2}\sqrt{\frac{t}{f}}.$$

So,

$$n_{SP} = \frac{1}{2}n_{oligopoly}.$$

So the oligopoly would open up an inefficient number of firms (double the SP's choice). Too much variety compared with the social optimum.

# 4 Lecture 4: February 21, 2006

## 4.1 Second Degree Price Discrimination: Non-linear Pricing

**Stole 1995 Paper**

- Consider anonymous non-linear pricing with a linear city of consumers. Firms locate at opposite ends of a line of length, $\Delta$. $\theta$ is the location of the consumer and

$$\theta \in [0, \Delta].$$

So $\theta_L = \theta$ is the distance from the consumer to the Left firm and:

$$\theta_R = \Delta - \theta,$$

is the distance to the Right firm.

- This model also has vertical product differentiation denoted by $v$. $v$ describes a consumer's taste for quality. Assume:

$$v \in [\underline{v}, \overline{v}].$$

- If a consumer buys a good from firm $i$ of quality/quantity $q$ at price $p$, her utility is:

$$U = u(q, \theta_i, v) - p,$$

where $u$ is in $C^1$, concave in $q$ and we normalize not buying from any firm to $u = 0$.

- Sorting Assumptions:

$$u_\theta < 0, \quad u_{\theta q} < 0, \quad u_v > 0, \quad u_{vq} > 0.$$

- Other Assumptions:

$$u_{\theta v} \leq 0, \quad u_{vv} \leq 0.$$

- The firm does not observe $\theta$ directly but does know the distribution: CDF, $F$, and PDF, $f$.

- Then:

$$f^L(\theta_L) = f(\theta_L),$$
$$f^R(\theta_R) = f(\Delta - \theta_R).$$

$F^L$ and $F^R$ have similar interpretations. This is rather confusing notation.

- Assume $v \sim G$ with density, $g$. Firm $i$'s per customer profit is:

$$\pi^i = p_i - c_i q_i.$$

- Assume for now that all consumers have the same $v$.

- **Definition** An indirect contract is one in which firms offer a menu of price/quality contracts, $[q_i, p_i(q_i)]$, and a consumer that picks $q_i$ must pay $p_i(q_i)$. Consumers sort themselves.

- **Definition** In a direct contract, each firm tailors the contracts to consumers of type $\theta$, ie: $[q_i(\theta), p_i(\theta)]$. The firm makes sure the consumers truthfully reveal their types by satisfying the IR and IC constraints. Again, consumers sort themselves.

- **Definition** Revelation Principal. If you have an INDIRECT contract, say of prices and quantities, which is offered by firms and achieves some outcome which satisfies IR and IC constraints, then this can also be achieved by a DIRECT revelation contract. Conversely, any outcome not achievable as an equilibrium in a direct mechanism, is also not achievable as an equilibrium using an indirect mechanism.

- Suppose firms offer the following contracts:

$$\{(q_i^*, p_i^*) \in Q_i x P_i | p_i^* = \rho_i(q_i^*)\},\ i \in \{L, R\}.$$

- These contracts produce an outcome where:
  - for all $\theta \in \Theta_L^*$, these consumers buy from firm $L$, and
  - for all $\theta \in \Theta_R^*$, these consumers buy from firm $R$.

  The prices and quantities choosen are:

$$(\tilde{q}_L(\theta), \tilde{p}_L(\theta)), \quad (\tilde{q}_R(\theta), \tilde{p}_R(\theta)).$$

- We need the prices and quantities choosen to be the solution to the consumer's optimization problem. Thus if $(\tilde{q}, \tilde{p})$ is an equilibrium, then:

$$\forall\ i \in \{R, L\},\ \forall\ \theta \in \Theta_i^*,\ \tilde{q}_i(\theta) \in arg\ max_{q \in Q_i}\ \{u(q, \theta, v) - \rho_i(q)\},$$

  and,

$$\tilde{p}_i(\theta) = \rho_i(\tilde{q}_i(\theta)).$$

  We also need:

$$Max_{q \in Q_i}\ \{u(q, \theta, v) - \rho_i(q)\} \geq Max\{Max\{\underbrace{u(q, \Delta - \theta, v) - \rho_j(q)}_{buying\ from\ other\ firm}\}, 0\}.$$

- If the above is satisfied, then there is an indirect mechanism that can achieve the same outcome. It must be the case that:

$$\forall\ i \in \{L, R\},\ \forall\ \theta \in \Theta_i^*,$$

$$IR: \quad \theta \in arg\ max_{\tilde{\theta} \in \Theta_i^*}\{u(\tilde{q}_i(\tilde{\theta}), \theta, v) - \tilde{p}_i(\tilde{\theta})\},$$

$$IC: \quad u(\tilde{q}_i(\theta), \theta, v) - \tilde{p}_i(\theta) \geq Max\{0, u(\tilde{q}_j(\Delta - \theta), \Delta - \theta, v) - \tilde{p}_j(\Delta - \theta)\}.$$

So these constraints say that there is no incentive to not reveal your true type and everyone is buying from the correct firm.

- So if can achieve an (equilibrium) outcome by a direct mechanism, then we can also achieve it indirectly. Thus, we might as well work with direct mechanisms. The only loss is that we may not observe the realistic contracts because many direct mechanisms may imply the same indirect mechanism.

- **Lemma**Note that a consumer of type $\theta_i$ who fakes it by announcing a type $\hat{\theta}_i$, has indirect utility:

$$U^i(\hat{\theta}_i, \theta_i) = u(q_i(\hat{\theta}_i), \theta_i, v) - p_i(\hat{\theta}_i),$$

and a truth-telling consumer has indirect utility:

$$U^i(\theta_i) = u(q_i(\theta_i), \theta_i, v) - p_i(\theta_i).$$

Then given firm $-i$'s truthtelling inducing contract, $(p_{-i}, q_{-i})$, a consumer buys from firm $i$ and reports $\theta_i$ IF AND ONLY IF $\theta_i \leq \overline{\theta}_i$ where:

  - (1) $U^i(\theta_i) = U^i(\overline{\theta}_i) - \int_{\theta_i}^{\overline{\theta}_i} u_\theta(q_i(s), s, v)ds$.
  - (2) $U^i(\overline{\theta}_i) \geq Max\{U^{-i}(\Delta - \overline{\theta}_i), 0\}$.
  - (3) $q_i$ is nonincreasing.

This lemma is trying to determine the cutoffs, $\overline{\theta}_L$ and $\overline{\theta}_R$ for each firm. There is no good intuition for the first condition, number 2 says the "worst type's" IR contraint must be satisified, ie the guy furthest away that still buys, and number 3 says that people further away from the firm must choose a lower quality/quantity.

**Proof of the Lemma**

- First we prove that if the IC and IR constraints are satisfied, then conditions (1), (2), and (3) are satified.

- The IC constraints imply the following about types $\theta_i$ and $\hat{\theta}_i$:

$$U^i(\theta_i) \geq U^i(\hat{\theta}_i, \theta_i) = u(q_i(\hat{\theta}_i), \theta_i, v) - p_i(\hat{\theta}_i),$$

$$U^i(\hat{\theta}_i) \geq U^i(\theta_i, \hat{\theta}_i) = u(q_i(\theta_i), \hat{\theta}_i, v) - p_i(\theta_i).$$

Truthtelling is optimal.

- Now subtract $U^i(\hat{\theta}_i)$ from the first expression and $U^i(\theta_i)$ from the second:

$$U^i(\theta_i) - U^i(\hat{\theta}_i) \geq u(q_i(\hat{\theta}_i), \theta_i, v) - p_i(\hat{\theta}_i) - [u(q_i(\hat{\theta}_i), \hat{\theta}_i, v) - p_i(\hat{\theta}_i)],$$

$$U^i(\hat{\theta}_i) - U^i(\theta_i) \geq u(q_i(\theta_i), \hat{\theta}_i, v) - p_i(\theta_i) - [u(q_i(\theta_i), \theta_i, v) - p_i(\theta_i)].$$

These expressions simplify to:

$$U^i(\theta_i) - U^i(\hat{\theta}_i) \geq u(q_i(\hat{\theta}_i), \theta_i, v) - u(q_i(\hat{\theta}_i), \hat{\theta}_i, v),$$

$$U^i(\hat{\theta}_i) - U^i(\theta_i) \geq u(q_i(\theta_i), \hat{\theta}_i, v) - u(q_i(\theta_i), \theta_i, v).$$

Multiply the second equation by $-1$ yields:

$$U^i(\theta_i) - U^i(\hat{\theta}_i) \leq u(q_i(\theta_i), \theta_i, v) - u(q_i(\theta_i), \hat{\theta}_i, v).$$

And combine this with the first:

$$u(q_i(\theta_i), \theta_i, v) - u(q_i(\theta_i), \hat{\theta}_i, v) \geq U^i(\theta_i) - U^i(\hat{\theta}_i) \geq u(q_i(\hat{\theta}_i), \theta_i, v) - u(q_i(\hat{\theta}_i), \hat{\theta}_i, v).$$

- Divide everything by $\theta_i - \hat{\theta}_i$:

$$\frac{u(q_i(\theta_i), \theta_i, v) - u(q_i(\theta_i), \hat{\theta}_i, v)}{\theta_i - \hat{\theta}_i} \geq \frac{U^i(\theta_i) - U^i(\hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \geq \frac{u(q_i(\hat{\theta}_i), \theta_i, v) - u(q_i(\hat{\theta}_i), \hat{\theta}_i, v)}{\theta_i - \hat{\theta}_i}.$$

So this looks damn close to a derivative. Since these conditions must hold for ALL $\theta$'s, they must hold in the limit as $\theta_i \to \hat{\theta}_i$. This means:

$$\frac{dU^i(\theta_i)}{d\theta_i} = u_\theta(q_i(\theta_i), \theta_i, v).$$

- Integrate both sides (by the FTC):

$$U^i(\theta_i) = U^i(\overline{\theta}_i) - \int_{\theta_i}^{\overline{\theta}_i} u_\theta(q_i(s), s, v) ds,$$

which is condition (1).

- We get condition (2) because if the IR holds, it clearly holds for the "worst-case" consumer.

- What about condition (3) ? Assume WLOG, $\theta_i > \hat{\theta}_i$, then from above, we have:

$$u(q_i(\theta_i), \theta_i, v) - u(q_i(\theta_i), \hat{\theta}_i, v) \geq u(q_i(\hat{\theta}_i), \theta_i, v) - u(q_i(\hat{\theta}_i), \hat{\theta}_i, v).$$

Applying the squeeze technique above (dividing by the difference in the thetas and taking a limit):

$$u_\theta(q_i(\theta_i), \theta_i, v) \geq u_\theta(q_i(\hat{\theta}_i), \hat{\theta}_i, v).$$

And finally, since we have $u_{\theta q} < 0$, this means $q_i(\theta_i) \leq q_i(\hat{\theta}_i)$ which proves $q$ is nonincreasing.

- So we've proved one direction. Now assume (1), (2) and (3) hold. Lets show that IR and IC hold. Clearly,

$$u(q_i(\hat{\theta}_i), \theta_i, v) - u(q_i(\hat{\theta}_i), \hat{\theta}_i, v) = \int_{\hat{\theta}_i}^{\theta_i} u_\theta(q_i(\hat{\theta}_i), s, v)ds.$$

Since $q$ is non-increasing and $u_{\theta q} < 0$,

$$\begin{aligned} u(q_i(\hat{\theta}_i), \theta_i, v) - u(q_i(\hat{\theta}_i), \hat{\theta}_i, v) &\leq \int_{\hat{\theta}_i}^{\theta_i} u_\theta(q_i(s), s, v)ds \\ &= U^i(\theta_i) - U^i(\hat{\theta}_i) \end{aligned}$$

- So this is just our IC constraint as before (tranformed a bit), while the IR holds because condition (2) says it is satisifed for the worse case guy, it must also hold for all consumers.

- QED. The Lemma is proved.

**Expected Profits**

- So what should we do with this lemma? Suppose 2 firms are offering the direct incentive contracts and we want to find a Nash equilibrium. Each firm, $i = L$ and $i = R$, solves:

$$Max\ \{E[\pi^i] = \int_0^{\overline{\theta}_i} [p_i(s) - c_i q_i(s)] f^i(s)ds\},$$

subject to,

$$IR\ and\ IC.$$

- Recall $U^i = u(\cdot) - p_i$, so we can write $p_i = u(\cdot) - U^i$, and plug this into our expected profits:

$$E[\pi^i] = \int_0^{\overline{\theta}_i} [u(q_i(s), s, v) - U^i(s) - c_i q_i(s)] f^i(s)ds.$$

- **Aside on Integration By Parts**. To go further, we will need to rewrite the integral term involving $U^i$. Consider condition (1) in the lemma:

$$U^i(\theta_i) = U^i(\overline{\theta}_i) - \int_{\theta_i}^{\overline{\theta}_i} u_\theta(q_i(s), s, v)ds.$$

Multiply by the density and integrate (and then simplify):

$$
\begin{aligned}
\int_0^{\overline{\theta}_i} U^i(s) f^i(s) ds &= \int_0^{\overline{\theta}_i} [U^i(\overline{\theta}_i) - \int_s^{\overline{\theta}_i} u_\theta(q_i(t), t, v) dt] f^i(s) ds \\
&= \int_0^{\overline{\theta}_i} U^i(\overline{\theta}_i) f^i(s) ds - \int_0^{\overline{\theta}_i} [\int_s^{\overline{\theta}_i} u_\theta(q_i(t), t, v) dt] f^i(s) ds \\
&= U^i(\overline{\theta}_i) F^i(\overline{\theta}_i) - \int_0^{\overline{\theta}_i} [\int_s^{\overline{\theta}_i} u_\theta(q_i(t), t, v) dt] f^i(s) ds
\end{aligned}
$$

Now integrate by parts, recall $\int u dv = uv - \int v du$. If

$$
u = \int_s^{\overline{\theta}_i} u_\theta(q_i(t), t, v) dt,
$$

then,

$$
du = \underbrace{u_\theta(q_i(\overline{\theta}_i), \overline{\theta}_i, v)}_{=0?} - u_\theta(q_i(s), s, v) ds = -u_\theta(q_i(s), s, v) ds.
$$

And:

$$
dv = f^i(s) ds,
$$

then:

$$
v = F^i(s).
$$

So back to our equation:

$$
\begin{aligned}
\int_0^{\overline{\theta}_i} U^i(s) f^i(s) ds &= U^i(\overline{\theta}_i) F^i(\overline{\theta}_i) - \int_0^{\overline{\theta}_i} \underbrace{[\int_s^{\overline{\theta}_i} u_\theta(q_i(t), t, v) dt]}_{u} \underbrace{f^i(s) ds}_{dv} \\
&= U^i(\overline{\theta}_i) F^i(\overline{\theta}_i) - \overbrace{\left[ \underbrace{\int_s^{\overline{\theta}_i} u_\theta(q_i(t), t, v) dt}_{u} \underbrace{F^i(s)}_{v} \right]_0^{\overline{\theta}_i}}^{=0} - \underbrace{\int_0^{\overline{\theta}_i} u_\theta(q_i(s), s, v) F^i(s) ds}_{\int v du} \\
&= U^i(\overline{\theta}_i) F^i(\overline{\theta}_i) - \int_0^{\overline{\theta}_i} u_\theta(q_i(s), s, v) F^i(s) ds \\
&= \int_0^{\overline{\theta}_i} U^i(\overline{\theta}_i) f^i(s) ds - \int_0^{\overline{\theta}_i} u_\theta(q_i(s), s, v) F^i(s) ds \\
&= \int_0^{\overline{\theta}_i} [U^i(\overline{\theta}_i) - u_\theta(q_i(s), s, v) \frac{F^i(s)}{f^i(s)}] f^i(s) ds \\
\int_0^{\overline{\theta}_i} U^i(s) f^i(s) ds &= \int_0^{\overline{\theta}_i} [U^i(\overline{\theta}_i) - u_\theta(q_i(s), s, v) \frac{F^i(s)}{f^i(s)}] f^i(s) ds
\end{aligned}
$$

This is the end of the aside.

- Now return to our expected profit function and substitute:

$$\begin{aligned}
E[\pi^i] &= \int_0^{\overline{\theta}_i} [u(q_i(s),s,v) - U^i(s) - c_i q_i(s)] f^i(s) ds \\
&= \int_0^{\overline{\theta}_i} [u(q_i(s),s,v) - c_i q_i(s)] f^i(s) ds - \int_0^{\overline{\theta}_i} U^i(s) f^i(s) ds \\
&= \int_0^{\overline{\theta}_i} [u(q_i(s),s,v) - c_i q_i(s)] f^i(s) ds - \int_0^{\overline{\theta}_i} [\underbrace{U^i(\overline{\theta}_i)}_{max\{U^{-i}(\Delta-\overline{\theta}_i),0\}} - u_\theta(q_i(s),s,v)\frac{F^i(s)}{f^i(s)}] f^i(s) ds \\
&= \int_0^{\overline{\theta}_i} [u(q_i(s),s,v) - c_i q_i(s) - max\{U^{-i}(\Delta-\overline{\theta}_i),0\} + u_\theta(q_i(s),s,v)\frac{F^i(s)}{f^i(s)}] f^i(s) ds
\end{aligned}$$

- Now take the FOC w.r.t. $q$:

$$[u_q(q_i(\theta),\theta,v) - c_i] f^i(\theta) + \underbrace{u_{\theta q}(q_i(\theta),\theta,v)}_{<0} \underbrace{F^i(\theta)}_{>0} = 0$$

This implies that for all $\theta_i > 0$,

$$u_q(q_i(\theta),\theta,v) - c_i > 0.$$

Or,

$$u_q(q_i(\theta),\theta,v) > c_i.$$

- Punchline: The marginal utility of quality/quantity exceeds the marginal cost of quality/quantity. There is a distortion away from the optimal level of quality. Note for the "best type", ie the guy located right on top of firm $i$, $\theta_i = 0$, so $F^i(0) = 0$ and:

$$u_q(q_i(0),0,v) = c_i,$$

optimal!

- So finally, what about this boundry condition, $\overline{\theta}$? We have two cases.
  - (1) $\overline{\theta}_L < \Delta - \overline{\theta}_R$. See G-4.1. In this case we have 2 local monopolies where there is no overlap. Indeed, it might be that there are customers in the "middle" which do not buy at all. The boundry condition would be defined by:

$$u(q_i(\overline{\theta}_i),\overline{\theta}_i,v) - c_i q_i(\overline{\theta}_i) + u_\theta(q_i(\overline{\theta}_i),\overline{\theta}_i,v)\frac{F^i(\overline{\theta}_i)}{f^i(\overline{\theta}_i)} = 0.$$

  So the expected profit of the guy furthest out is zero for both firms.

– (2) $\overline{\theta}_L > \Delta - \overline{\theta}_R$. Now we have overlap and we can define the marginal man by:

$$U^L(\overline{\theta}_L) = U^R(\Delta - \overline{\theta}_L) = \overline{U}.$$

Thus for $i = \{L, R\}$, (somehow!):

$$u(q_i(\overline{\theta}_i), \overline{\theta}_i, v) - c_i q_i(\overline{\theta}_i) - \overline{U} = 0.$$

Again the profits on the boundry guy are zero. See G-4.2.

# Problem Set 1

## 1. Linear Highway Model

- Part (a). Monopolist faces demand:

  $$V - tx - p = 0.$$

  Or,

  $$x = \frac{V - p}{t}.$$

  So demand is:

  $$D(p) = min\{1, \frac{V - p}{t}\}.$$

  Profits:

  $$\pi = p * (\frac{V - p}{t}).$$

  FOC:

  $$\frac{d\pi}{dp} = \frac{V - p}{t} - \frac{p}{t} = 0 \Longrightarrow p = V/2.$$

  So if demand is equal to 1, it must be that:

  $$1 = D(p) \Leftrightarrow 1 = \frac{V - V/2}{t} \Longrightarrow V = 2t.$$

  So as long as $V \geq 2t$, the monopolist will serve the whole market.

- Part (b). If $x^* \in [0, 0.5]$, need $p^*$ to be ok with the guy at 1, so:

  $$V - t(1 - x^*) - p^* = 0,$$

  $$p^* = V - t(1 - x^*).$$

  If $x^* \in [0.5, 1]$, need $p^*$ to be ok with the guy at 0, so:

  $$V - tx^* - p^* = 0.$$

  $$p^* = V - tx^*.$$

  So,

  $$p^* = V - t * max\{1 - x^*, x^*\}.$$

- Part (c). Assume $V = t$. Monopolist located at 0. Fixed costs of opening at 1 is $f$. For what values of $f$ does the monopolist open? For what values does the SP open? With only a store at 0, selling to the whole market means:

  $$V - t - p = 0 \Longrightarrow p = 0,$$

so the monopolist will find it better to sell to only part of the market. As above,

$$p^* = V/2.$$

Profits are thus:

$$\pi = V/2 * \frac{V - V/2}{V} = \frac{V}{4}.$$

Opening a second store at location 1 would yield additional profits equal to $\frac{V}{4}$, just because $V = t$ so the monopolist does not canabalize his first store by opening the second. So the monopolist opens if:

$$f < \frac{1}{4}V.$$

What about the social planner? Opening the store at 1 generates new CS of $\frac{1}{2} * V$. The guy located at 1 with a price of $V/2$ would have surplus: $V - V(1-1) - V/2 = V/2$. The transportation costs $\frac{1}{2} * \frac{1}{2} * \frac{1}{2} * V$, ie the area of a triangle with base 1/2 and height $V/2$. So the social surplus from opening is:

$$\frac{V}{2} - \frac{1}{2}\frac{1}{2}\frac{V}{2} - f = \frac{3}{8}V - f.$$

So the SP opens if:

$$f < \frac{3}{8}V.$$

In this example, the monopolist undersupplies variety.

- Part (d). So $t = 2$ for consumers $x \in [0, 1/4]$ and $t = 1$ for consumers $x \in [1/4, 1]$. $V = 5$. Where would the monopolist locate a single store. Where would the social planner locate? $V$ is high enough for the monopolist to want to sell to all consumers. The optimal location will be one in which the transport costs are the same for the guy at 0 and the guy at 1. Otherwise, the monopolist would want to move towards the higher of the two because he could raise his price. So we need:

$$2x = 1(1 - x)$$

$$x^* = \frac{1}{3},$$

is optimal for the monopolist. The social planner is trying to minimize transport costs. So:

$$Min\ C(x^*) = \int_0^{1/4} 2 * |x - x^*| dx + \int_{1/4}^1 |x - x^*| dx.$$

Or, since with constant transport costs, the SP would choose the center of the interval, by increasing the costs for those on the left, this will shift the optimal location to the left, yielding an $x^*$ somewhere between 1/4 and 1/2. Thus,

$$Min\ C(x^*) = \int_0^{1/4} 2 * (x^* - x)dx + \int_{1/4}^{x^*} (x^* - x)dx + \int_{x^*}^{1} (x - x^*)dx.$$

$$x^* = \frac{3}{8}.$$

So the monopolist who wants to serve everyone, the optimal location only depends on the transport costs of the edge consumers, not the kink point.

## 2. Hexagonal Spacial Differentiation

- For the transportation cost minimizing SP, the node of the hexagon where $t = 2$ is clearly the minimizing point. For a monopolist who sells to all (which is shown to be optimal), locating at the point where $t = 2$, yields:

$$5 - 3 * \frac{1}{6} - p^* \geq 0 \Rightarrow p^* = 4.5.$$

If he locates at one of the points adjacent to the $t = 2$ consumers, then two constraints must be satisfied:

$$5 - 2 * \frac{1}{6} - p^* \geq 0 \Rightarrow p^* \geq 4.67.$$

$$5 - 3 * \frac{1}{6} - p^* \geq 0 \Rightarrow p^* \geq 4.5.$$

So again $p^* = 4.5$. This is also true for the other adjacent point to $t = 2$. Consider locating 2 points away from the $t = 2$ consumers. Then:

$$5 - 2 * 2 * \frac{1}{6} - p^* \geq 0 \Rightarrow p^* \geq 4.33.$$

$$5 - 3 * \frac{1}{6} - p^* \geq 0 \Rightarrow p^* \geq 4.5.$$

So $p^* = 4.33$, yielding lower profits. So a monopolist would locate at either the $t = 2$ node, or either of those nodes directly adjacent to the $t = 2$ consumers.

## 3. Insurance

- $\theta \in [0, 1/2]$ with $\theta = \text{Prob(Loss)}$. Loss $= L$. Assume:

$$F(\theta) = 2\theta, \ \ \theta \in [0, 1/2].$$

Assume consumers are willing to pay $V(\theta) = \alpha\theta$ for insurance to cover the potential loss $L$. Assume consumers risk averse so $\alpha > L$.

- So we seek some $\theta^*$ such that for $\theta \in [\theta^*, 1/2]$, consumers buy, and otherwise they do not. The expected cost to the insurance company is:

$$
\begin{aligned}
E[C(\theta^*)] &= \frac{\int_{\theta^*}^{1/2} L\theta f(\theta) d\theta}{\int_{\theta^*}^{1/2} f(\theta) d\theta} \\
&= 2L \frac{\int_{\theta^*}^{1/2} \theta d\theta}{2\int_{\theta^*}^{1/2} d\theta} \\
&= L \frac{[0.5\theta^2]_{\theta^*}^{1/2}}{[\theta]_{\theta^*}^{1/2}} \\
&= (1/2)L \frac{\frac{1}{4} - \theta^{*2}}{\frac{1}{2} - \theta^*} \\
&= (1/2)L \frac{(1/2 - \theta^*)(1/2 + \theta^*)}{(\frac{1}{2} - \theta^*)} \\
&= \frac{L(1/2 + \theta^*)}{2}
\end{aligned}
$$

If everyone from $\theta^*$ on up buys, then we need:

$$
\begin{aligned}
V(\theta^*) &= E[C(\theta^*)] \\
\alpha\theta^* &= \frac{L(1/2 + \theta^*)}{2} \\
\alpha\theta^* &= \frac{L}{4} + \frac{L\theta^*}{2} \\
\theta^*(\alpha - L/2) &= \frac{L}{4} \\
\theta^* &= \frac{L/4}{\alpha - L/2} \\
\theta^* &= \frac{L}{4\alpha - 2L}
\end{aligned}
$$

And this defines our cutoff.

# 5 Lecture 5: February 28, 2006

## 5.1 Remark on the Linear City Model

- We assumed in the linear city that pricing was uniform across consumers. That may not be a realistic assumption. In most markets, prices often vary across consumers for a number of different reasons.
- Consider a consumer located at $\theta^* \in [0,1]$. Her payoff equals $V - P-$ Transportation costs.
- Suppose the consumer held an auction for her business so the two firms located at each end of the city had to bid for her to buy from them. This induces a Bertrand type of competition with the winner's margin determined by the point when the loser's price hits his marginal cost. It works out that this margin is the different between the two firm's transportation costs.

## 5.2 Consumer Switching Costs and Search

**Klemperer (1987)**

- Consider two firms, $A$ and $B$, which sell to a set of consumers in two periods, 1 and 2.
- Inverse demand is $f(q)$ which induces demand:

$$q = h(p) = f^{-1}(q).$$

- Let $\sigma^i, i = \{A, B\}$, be the proportion of the consumers in period 1 that buy from each firm. Note $\sigma^A + \sigma^B = 1$.
- In period 2, consumers may switch between products at cost.
- Assume $\Gamma(w)$ is the proportion of the firm's consumers with switching costs less than $w$. Thus, $\gamma(w) = d\Gamma/dw$ is our density. Assume,

$$\Gamma(0) = 0,$$

  so no one has negative switching costs, but $\gamma(0)$ is not necessarily 0, ie, some consumers may have zero switching costs.
- Assume WLOG, the prices set in stage 2 satisfy:

$$p^A \leq p^B.$$

  So we will only get consumers switching away from B towards A.

- Demand for firm A:

$$q^A = \underbrace{\sigma^A h(p^A)}_{1} + \underbrace{\sigma^B \Gamma(p^B - p^A) h(p^B)}_{2} + \underbrace{\sigma^B \int_{p^A}^{p^B} \Gamma(r - p^A)[-d\ h(r)]}_{3}.$$

So term 1 is all of firm A's consumers who continue to buy from A. Term 2 are those consumers who would have bought from B but have switching costs which are less than the savings they would get from switching to the lower priced firm A so they switch. Finally term 3 are those consumers who would NOT have bought from B but have small enough switching costs to move to A.

- Demand for firm B:

$$q^B = \text{The Non-Switchers} = \sigma^B[1 - \Gamma(p^B - p^A)]h(p^B).$$

- Consider differentiating $q^A$ with respect to $p^A$. We'll need this term in a moment:

$$\frac{dq^A}{dp^A} = \sigma^A h'(p^A) - \sigma^B \gamma(p^B - p^A) h(p^B) - \sigma^B \int_{p^A}^{p^B} \gamma(r - p^A)[-d\ h(r)].$$

- We now seek the non-cooperative (Nash) equilibrium. Given a profit function for firm A: $\pi^A = p^A q^A - c^A(q^A)$, the FOC is:

$$\begin{aligned} 0 &= \frac{\partial \pi^A}{\partial p^A} = q^A + p^A \frac{\partial q^A}{\partial p^A} - \frac{\partial c^A}{\partial q^A}\frac{\partial q^A}{\partial p^A} \\ &= q^A + [p^A - \frac{\partial c^A}{\partial q^A}]\frac{\partial q^A}{\partial p^A} \\ &= \sigma^A h(p^A) + \sigma^B \Gamma(p^B - p^A) h(p^B) + \sigma^B \int_{p^A}^{p^B} \Gamma(r - p^A)[-d\ h(r)] \\ &\quad + [p^A - \frac{\partial c^A}{\partial q^A}][\sigma^A h'(p^A) - \sigma^B \gamma(p^B - p^A) h(p^B) - \sigma^B \int_{p^A}^{p^B} \gamma(r - p^A)[-d\ h(r)]] \end{aligned}$$

So what should we do with this beast ? First assume $c^A = c^B$ and $\sigma^A = \sigma^B = \frac{1}{2}$. Thus costs are the same and firm's split the market equally. There may be other equilibria, but one of interest is the symmetric case where $p^A = p^B = p$. So firms in period 2 find it optimal to set the same price.

- The FOC simplifies to (dividing through by $\sigma$):

$$h(p) + [p - \frac{\partial c}{\partial q^A}][h'(p) - \gamma(0)h(p)].$$

- Consider two cases:

- (1) $\gamma(0) = 0$. FOC further simplifies to:

$$h(p) + [p - \frac{\partial c}{\partial q^A}]h'(p),$$

or,

$$p = \frac{(\partial c/\partial q) * h'(p) - h(p)}{h'(p)},$$

the Monopolist's Markup! So if no one has zero switching costs, on the margin you don't capture any consumers so we have driven away all the competition.

- (2) $\gamma(0) \to \infty$. For the FOC to be satisfied, it must be that $p \to \frac{\partial c}{\partial q^A}$, or price equals marginal cost! So we move towards perfect competition.

- So $\gamma(0)$ is key. As $\gamma(0)$ gets larger, $\Gamma(w)$ becomes steeper at $w = 0$. The larger is the slope of the CDF at $w = 0$, the more competitive the market will be.

- Some industries with switching costs might be computers, software, cell phones, etc. It seems likely that we should observe firms pricing at a discount in period 1 to lock in the consumers in period 2. Indeed, Klemperer shows this in the following simple model.

- Assume $v$ is some strategic variable under the firm's control, say price. In a 2 period setting, firm profits are:

$$\pi = \pi_1(v) + \lambda\pi_2(\sigma(v)),$$

where $\lambda$ is a discount factor and $\sigma(v)$ represents the proportion of consumers the firm secures in period 1 by choosing $v$.

- Assuming:

$$\frac{\partial \sigma}{\partial v} > 0,$$

differentiating our profit function:

$$\frac{\partial \pi}{\partial v} = \frac{\partial \pi_1}{\partial v} + \lambda \underbrace{\frac{\partial \pi_2}{\partial \sigma}}_{+ve} \underbrace{\frac{\partial \sigma}{\partial v}}_{+ve} = 0.$$

This implies:

$$\frac{\partial \pi_1}{\partial v} < 0,$$

or firms are pricing aggressively in the first period to lock in consumers. Eg, they set a lower price than would otherwise be optimal in the absence of switching costs.

**Farrell and Gullani**

- Consider a monopolist in a 2 period model with constant marginal cost, $c$. There is a common discount factor, $\delta$.

- If consumers are to use the product, they have to pay a one time set up cost, $F$ (installation costs, human capital development, etc).
- Consumer's willingness to pay each period is $V$ and they have discrete demand of zero or one unit.
- Assume there are positive gains from trade, ie:

$$(V - c) + \delta(V - c) - F > 0.$$

The monopolist may or may not be able to commit to a set of prices in each of the two periods.

- **Commitment Case** Assume the monopolist commits to prices $(p_1, p_2)$. For consumers to buy in BOTH periods, a few things must happen:

$$V(1+\delta) - p_1 - \delta p_2 - F \geq 0 \quad (1)$$

$$V(1+\delta) - p_1 - \delta p_2 - F \geq V - p_1 - F \quad (2)$$

$$V(1+\delta) - p_1 - \delta p_2 - F \geq \delta(V - p_2 - F) \quad (3)$$

So this says that buying in both periods is better (1) than not buying at all, (2) than buying only in period 1, and (3) than buying only in period 2. To extract all consumer surplus, the firm sets prices such that (1) holds with equality and (2) and (3) are satisfied. One set of prices that satisfies this is $p_1 = V - F$ and $p_2 = V$. This implies:

$$(1) \implies V(1+\delta) - (V - F) - \delta(V) - F = 0.$$

$$(2) \implies V(1+\delta) - (V - F) - \delta(V) - F \geq V - (V - F) - F \Leftrightarrow 0 \geq 0.$$

$$(3) \implies V(1+\delta) - (V - F) - \delta(V) - F \geq \delta(V - (V) - F) \Leftrightarrow 0 \geq -\delta F.$$

- **Non-commitment Case** Assuming the consumer invests in the first period, $p_2 = V$ is the best the firm can do in period 2 to extract all surplus. However there is no guarantee that $p_1 = V - F$ is even positive! If $F > V$, we may not be able to implement the prices in the commitment case. If $p_1 = 0$, the smallest it can be, and $p_2 = V$, condition (1) may not be satisfied. *Alternative:* Suppose the firm offers a per period royalty rate to any firm that wants to use its technology to produce the same product in period 2. In particular, if a royalty rate is:

$$R = V - (1+\delta)^{-1}F - c,$$

then,

$$p_1 = p_2 = R + c = V - (1+\delta)^{-1}F.$$

Lets verify that all our conditions are now satisfied:

$$
\begin{aligned}
(1) \quad &\Longrightarrow \quad V(1+\delta) - (V - (1+\delta)^{-1}F) - \delta(V - (1+\delta)^{-1}F) - F = 0 \\
&\Leftrightarrow \quad V + V\delta - V + F(1+\delta)^{-1} - \delta V + \delta F(1+\delta)^{-1} - F = 0 \\
&\Leftrightarrow \quad F(1+\delta)^{-1} + \delta F(1+\delta)^{-1} - F = 0 \\
&\Leftrightarrow \quad \frac{F + \delta F - F - \delta F}{1+\delta} = 0 \\
&\qquad \text{good}
\end{aligned}
$$

$$
\begin{aligned}
(2) \quad &\Longrightarrow \quad V(1+\delta) - (V - (1+\delta)^{-1}F) - \delta(V - (1+\delta)^{-1}F) - F \\
&\qquad \geq V - (V - (1+\delta)^{-1}F) - F \\
&\Leftrightarrow \quad V + V\delta - V + (1+\delta)^{-1}F - \delta V + \delta F(1+\delta)^{-1} - F \geq V - V + (1+\delta)^{-1}F - F \\
&\Leftrightarrow \quad (1+\delta)^{-1}F + \delta F(1+\delta)^{-1} - F \geq (1+\delta)^{-1}F - F \\
&\Leftrightarrow \quad 0 \geq -\frac{F\delta}{1+\delta} \\
&\qquad \text{good}
\end{aligned}
$$

$$
\begin{aligned}
(3) \quad &\Longrightarrow \quad V(1+\delta) - (V - (1+\delta)^{-1}F) - \delta(V - (1+\delta)^{-1}F) - F \\
&\qquad \geq \delta(V - (V - (1+\delta)^{-1}F) - F) \\
&\Leftrightarrow \quad 0 \geq \delta(V - V + (1+\delta)^{-1}F - F) \\
&\Leftrightarrow \quad 0 \geq -\frac{F\delta^2}{1+\delta} \\
&\qquad \text{good}
\end{aligned}
$$

- So we create a competitive environment in the second period through the royalty rate and this solves the inconsistency problem we had above when $F > V$.
- Consumers are ensured they will not be "held up" in the second period.

# 6 Lecture 6: March 7, 2006

## 6.1 Search and Switching Costs - Dudey (1990 AER)

- Question: why do all the music shops in Portland, OR locate around the same few blocks of town? Dudey has an answer.
- Model Timing:
  - Stage 1: Firms simultaneously choose a location from a discrete set. Multiple firms may choose the same point.
  - Stage 2: $m$ consumers learn locations and decide where to shop. Can only choose one location.
  - Stage 3: Firms choose quantities (Cournot competition).
  - Stage 4: Consumers make purchasing decisions.

  Note that if NO firms co-locate, all firms have a monopoly over the consumers at their location.
- Demand, $q = f(p)$, satisfies,

$$\forall\, p' \ni f(p') > 0, \;\; f(p) \geq f(p'),\; \forall\, p < p',$$

  ie demand is decreasing below the choke price. Also,

$$\int_{-\infty}^{\infty} Max\,\{f(p), 0\} dp < \infty,$$

  ie the area under the demand curve is finite. Let $p = f^{-1}(q) \equiv g(q)$.
- How does the market work? Let $x$ be the number of consumers at a location. If firms at that location produce less than $x * f(0)$, then consumers make purchases at the price that clears the market. Otherwise, $p = 0$. This assumption (firms that produce so much that price is driven to zero) is probably not necessary for the rest of the analysis.
- **Definitions**

  $m$: number of consumers
  $n$: number of firms
  $f(p)$: individual's demand
  $c$: all firm's constant marginal cost
  $q^c(x, y)$: cournot equilibrium per firm quantity at a location with $x$ consumers and $y$ firms
  $\pi(x, y)$: cournot equilibrium per firm profits at a location with $x$ consumers and $y$ firms

- **Assumptions**

- (1) $\forall\, x, y > 0$, $\pi(x, y) > 0$.
- (2) $\forall\, x > 0$, $\pi(x, y)$ is decreasing in $y$.
- (3) $\forall\, x > 0$, $\underbrace{yq^c(x, y)}_{Q}$ is increasing in $y$.

- **Proposition** There exists an equilibrium in which ALL firms locate at the same location.

- **Lemma** The Cournot equilibrium price at any location that attracts at least one consumer does NOT depend on the number of consumers at that location. In addition, the Cournot equilibrium quantity and per firm profit is linearly homogenous in the number of consumers.

- **Proof of Lemma** Suppose $q^c(x, y)$ is an equilibrium quantity. Then $\forall\, q \geq 0$,

$$\underbrace{\Big[\overbrace{g\Big(\frac{yq^c(x,y)}{x}\Big)}^{Price} - c\Big] q^c(x,y)}_{per\ firm\ profits} \geq \underbrace{\Big[g\Big(\frac{q}{x} + \frac{(y-1)q^c(x,y)}{x}\Big) - c\Big] q}_{deviation\ to\ q}. \quad (1)$$

This holds for all $x$, so consider the condition for a location with only one consumer, $x = 1$:

$$\Big[g\Big(yq^c(1,y)\Big) - c\Big] q^c(1,y) \geq \Big[g\Big(q + (y-1)q^c(1,y)\Big) - c\Big] q. \quad (2)$$

So now if the equilibrium quantity is linearly homogeneous, we have to have:

$$q^c(x, y) = xq^c(1, y). \quad (3)$$

So substitute (3) into (1):

$$\Big[g\Big(yq^c(1,y)\Big) - c\Big] xq^c(1,y) \geq \Big[g\Big(\frac{q}{x} + (y-1)q^c(1,y)\Big) - c\Big] q$$

$$\Big[g\Big(yq^c(1,y)\Big) - c\Big] q^c(1,y) \geq \Big[g\Big(\frac{q}{x} + (y-1)q^c(1,y)\Big) - c\Big] \frac{q}{x}. \quad (4)$$

But now (4) is exactly (2) except we have replaced the $q$ in (2) with $q/x$ in (4). Since (2) held for ALL deviations, $q/x$ is fine too, so (4) must hold. Thus we have shown that the cournot equilibrium quantity is indeed linearly homogenous (ie (3) holds).

So the equilibrium price can be expressed as:

$$p^* = g\Big(yq^c(1,y)\Big),$$

which does NOT depend on $x$! So the price is independent of the number of consumers at a given location. Also,

$$\pi(x, y) = [g(yq^c(1, y) - c]xq^c(1, y) = x\pi(1, y),$$

or the profit function is also linearly homogeneous in the number of consumers. QED on the lemma.

- **Proof of Proposition** Suppose all firms locate at location $S$. Suppose $n \geq 2$ and consumers shop at the location with the most firms or if there is a tie, then they choose $S$ if $S$ is among the most populated locations (in terms of firms). Clearly there is NO optimal deviation by any given firm because they would get no consumers by deviating to another location. Consumers also are playing optimally since they get the lowest price where the competition is fiercest. (Note although price does not depend on $x$, it does depend on $y$.) QED.

## 6.2 Two Sided Markets

- First some background. A necessary (though not sufficient) condition for a two-sided market is for a firm to provide services to two groups who transact with each other ONLY through the firm and not directly with each other.
- Some *possible* two-sided markets:
  - (1) eBay: (Buyers / Sellers).
  - (2) Financial markets: (Liquidity takers / Liquidity makers).
  - (3) Median companies like TV networks, newspapers, etc: (Reader or viewers / Advertisers).
  - (4) Credit card companies or "payment card systems: (Consumers / Merchants).
  - (5) Video game platforms: (Players / Game makers).
  - (6) Realtors: (Home buyers / Home sellers).
  - (7) In general, any firm with an input supplier who also sells to consumers.
- **Definition** A two-sided market is one where outcomes depend on the *price structure*, not just the overall *price level.*
- Now we apply this definition to some of our candidate markets and see if they hold up.
  - (1) For eBay, consider charging one dollar only to the seller of an item and nothing to the buyer, versus charging fifty cents to each. If the buyer and seller agree on a price of ten dollars, then they could negotiate this price differently depending on the price structure set by eBay to obtain exactly the same outcome. Because the buyer and seller transact directly with each other, this possibility of changing the price level of the item in response to the change in eBay's pricing structure is what makes eBay fail (at least by our definition) to be two-sided.

  - (3) Since there is no clear transaction taking place directly between television views and advertisers, this is a two-sided market.
  - (4) As long as merchants cannot charge different prices for different methods of payments (which they usually don't do), this is a (classic) two-sided market.
  - (5) Probably not because the price game makers charge to consumers could be adjusted depending on the pricing structure of the platform provider.
  - (6) Not two-sided.

- We'll lay out a model next time.

# 7 Lecture 7: March 14, 2006

## 7.1 More on Two-Sided Markets

**Rochet and Tirole**

- Consider a monopolist platform (like a monopolist credit card company) and assume there are two types of consumers: buyers and sellers (like restaurant patrons and restaurant owners) Let $i$ index consumers with $i \in \{B, S\}$.

- For the monopolist, he faces a constant marginal cost per transaction of $c$ and a cost per member consumer of $C^i$ which may vary for buyers and sellers.

- Consumers (credit card users and merchants) have an average benefit per transaction of $b^i$ and an overall benefit of membership of $B^i$.

- The number of (possible) transactions is $N^B * N^S$. That is, if there are 10 restaurant patrons and 5 restaurants, there are 50 possible transactions.

- Let $a^i$ be the price per transaction and $A^i$ is the membership dues to the platform for a consumer of type $i$. The net utility of a "buyer":

  $$U^B = (b^B - a^B)N^S + B^B - A^B,$$

  and for a "seller":

  $$U^S = (b^S - a^S)N^B + B^S - A^S.$$

- Though it is unclear in the paper, $b$ and $B$ are random variables with some assumed distribution.

- Then denote:

  $$N^i = Pr\{U^i \geq 0\}.$$

  So in determining the number of agents in each group we look at their net utility and if it's positive, we count them. So it's like we have normalized the number of potential agents on each side to 1.

- Define:

  $$p^i = a^i + \frac{A^i - C^i}{N^j},$$

  which is the average per transaction price.

- Thus demand can be written:

$$\begin{aligned} N^i = D^i(p^i, N^j) &= Pr\{U^i \geq 0\} \\ &= Pr\{(b^i - a^i)N^j + B^i - A^i \geq 0\} \\ &= Pr\{-a^i + b^i + \frac{B^i - A^i}{N^j} \geq 0\} \\ &= Pr\{b^i + B^i/N^j - \frac{C^i}{N^j} \geq a^i + \frac{A^i}{N^j} - \frac{C^i}{N^j}\} \\ &= Pr\{b^i + \frac{B^i - C^i}{N^j} \geq p^i\} \end{aligned}$$

- So we have written the demands as a function of the prices, but also $N^j$, the other side's agents. If we solve the two equations simultaneously, we can eliminate this variable and solve for:

$$N^B = n^B(p^B, p^S), \quad N^S = n^S(p^B, p^S).$$

- We can write these demands as:

$$n^B(p^B, p^S) = D^B(p^B, n^S(p^B, p^S)).$$

$$n^S(p^B, p^S) = D^S(p^S, n^B(p^B, p^S)).$$

- Differentating these expression w.r.t. $p^B$ yields:

$$\frac{\partial n^B}{\partial p^B} = \frac{\partial D^B}{\partial p^B} + \frac{\partial D^B}{\partial N^S}\frac{\partial N^S}{\partial p^B}.$$

$$\frac{\partial n^S}{\partial p^B} = \frac{\partial D^S}{\partial N^B}\frac{\partial N^B}{\partial p^B}.$$

- And solving (check paper for these):

$$\frac{\partial n^B}{\partial p^B} = \frac{\partial D^B/\partial p^B}{1 - (\partial D^B/\partial N^S)(\partial D^S/\partial N^B)}$$

$$\frac{\partial n^S}{\partial p^B} = \frac{(\partial D^B/\partial p^B)(\partial D^S/\partial N^B)}{1 - (\partial D^B/\partial N^S)(\partial D^S/\partial N^B)}$$

- So the monopolist's profit is:

$$\pi = (A^B - C^B)N^B + (A^S - C^S)N^S + (a^B + a^S - c)N^B N^S$$

$$= (p^B + p^S - c) * n^B(p^B, p^S)n^S(p^B, p^S).$$

- So the monopolist wants to choose $p^B$ and $p^S$ to maximize profits and we can do this in two stages. First we find the price structure, $p = p^B + p^S$, that maximizes the total trading volume. Then we'll find the optimal price level, $p$.
  - Stage 1:
    $$V(p) = max_{p^B,p^S} \{n^B(p^B, p^S)n^S(p^B, P^S)|p = p^B + p^S\},$$
    where $V$ is the maximized trading volume. Lagrangian:
    $$\mathcal{L} = n^B(p^B, p^S)n^S(p^B, P^S) - \lambda[p^B + p^S - p].$$
    Take the FOC and solve for the optimal prices given $p$.
  - Stage 2:
    $$Max_p \{(p - c)V(p)\}.$$
    FOC:
    $$V(p) + (p - c)V'(p) = 0 \Rightarrow \frac{p - c}{p} = -\frac{V(p)}{V'(p)p} = \frac{1}{\eta},$$
    where:
    $$\eta = -\frac{p}{V(p)}\frac{dV(p)}{dp},$$
    the price elasticity of demand.

  So this stage 2 maximization leads to something like the monopolist's markup equation.

- Lets move to something less confusing.

**Vincent and Schwartz (2006)**

- How does the imposition of a "no-surcharge rule" (NSC) effect the number of consumers of different types and their welfare?
- The NSC means that cash and credit card (CC) customers must be charged the same price.
- Suppose there are two types of consumers, CC users, indexed by $e$, who have mass 1, and Cash users, indexed by $c$, with mass $\alpha$.
- Assume consumers do NOT change their types.
- Preferences:
  $$\text{Cash Users: } U(p_c, q_c) = V(q_c) - p_c q_c,$$
  $$\text{Credit Users: } U(p_e, q_e) = V(q_e) - p_e q_e.$$
  Note $q_i$ is the per capita number of transactions for a customer of type $i$. Assume $V' > 0$, $V'' < 0$.

- Let $p_c$ be the cash price charged by the merchant. And let:

$$p_e = p_e^m + t,$$

be the price the CC customer pays, split between the merchant and the CC company respectively. Note that $t$ could be negative (discover cash back, frequent flyer miles, etc).

- Merchants are local monopolists with zero marginal costs. For CC transactions, merchants realize a benefit of $b \geq 0$ per transaction.

- Merchants pay a charge of $i$ (and interchange fee) to the Electronic Payment Network (EPN) per transaction.

- Thus the merchant's profit is:

$$\pi = \underbrace{\alpha p_c q_c}_{cash} + \underbrace{p_e^m q_e - (i-b) q_e}_{cc}.$$

- If you solve for the demands by differentiating the utility functions above with respect to quantities, you get:

$$V'(q_c) = p_c,$$
$$V'(q_e) = p_e = p_e^m + t.$$

- So the merchant's problem for CC users is to solve:

$$Max_x \{[p_e^m x - (i-b)x]\} = Max_x \{[V'(x) - (i+t-b)]x\},$$

where $x$ is the quantity chosen. This yields:

$$x_e^* = x(i+t-b),$$

ie, the optimal quantity is a function of the net EPN tax.

- The merchant's problem for cash users is:

$$Max_x \{[\alpha p_c x]\} = Max_x \{[V'(x)]x\},$$

where $x$ is the quantity chosen. This yields:

$$x_c^* = x_0.$$

- Clearly the price structure (ie, how $i$ compares to $t$) will not effect the way the merchant solves his problem. The same $x_e^*$ will result. Thus the Tirole definition of a two-sided market holds up.

- Denote $\pi^m(i,t,b)$ to be the value of the merchant's optimization problem. Clearly, if it is optimal for the merchant to offer both cash and credit as options, it must be that an IR constraint is satisfied:

$$\pi^m(i, t, b) \geq \alpha V'(x_0)x_0.$$

- So consider the merchant's problem again for CC users. The FOC is:

$$V'(x) - (i + t - b) + xV''(x) = 0.$$

Or,

$$i + t = xV''(x) + V'(x) + b,$$

which is the total amount extracted by the EPN per transaction.

- The EPN's problem is thus:

$$Max_x \{\underbrace{[xV''(x) + V'(x) + b]}_{i+t}x\}.$$

- **Proposition 1** Suppose surcharging is allowed. Then the equilibrium CC transactions, merchant profit, and EPN profits all depend only on EPN's total fee, $i + t$, and not on them individually. (As was shown above).

- **Assumptions and Properties**
  - (A1) $V'(x) = 1 - x$, $b < 1$.
  - (P1) Merchant's revenue is strictly concave in quantity and price, and any increase in the merchant's MC in serving cards is not fully passed through when surcharging is possible.
  - (P2) EPN's revenues, $(xV''(x) + V'(x) + b)x$, is strictly concave in $x$.
  - (P3) With surcharging, $i + t > b$.

- Under the NSR, $p_e^m \leq p_c$. Ie, the merchant cannot overcharge the CC users. This implies (plugging in from above):

$$V'(q_e) - t \leq V'(q_c).$$

- **Proposition 2** For a given $i + t = k > b$, define:

$$t^*(k) = V'(x^*(k - b)) - V'(x_0) > 0.$$

Then for any $(i, t)$ with $i + t = k$, and $t < t^*(k)$,

  - (1) With surcharging, $p_e^m > p_c$.
  - (2) With NSR (holding $i$ and $t$ fixed), cash purchases fall and CC purchases and EPN profits both rise.
  - (3) A cut in $t$ and an equivalent increase in $i$ (so holding $i + t$ fixed) increases per capita CC transactions and EPN profits.

- The proof for proposition 2 is best seen graphically. See G-7.1. The point R, at $x_0$ and $x^*(k-b)$, defines the optimal quantities of cash and CC purchases when surcharging is allowed. the indifference curves move outwards in concentric circles as shown. The NSR implies:

$$\begin{aligned} & p_e^m \leq p_c \\ \Leftrightarrow\ & V'(q_e) - t \leq V'(q_c) \\ \Leftrightarrow\ & (1-q_e) - t \leq (1-q_c) \\ \Leftrightarrow\ & q_c - q_e - t \leq 0 \\ \Leftrightarrow\ & q_e \geq q_c - t \end{aligned}$$

  This contstraint is shown in G-7.1 and is satisfied for the shaded region. Clearly R is outside of this region due to the conditions in proposition 2. Thus, the point E is the best the merchant can do. Now consider part (3) of the proposition. An increase in $t$ (and equivalent decrease in $i$) shifts the boundry of the NSR to the northwest. Clearly CC transactions rise and cash transactions fall as stated. Now recall the IR constraint when it binds:

$$\pi^m(i,t,b) = \alpha V'(x_0)x_0.$$

  The EPN will choose $t$ such that this constraint binds. This means that the boundry will shift out until we get something like G-7.2, the point where the constraint is just tangent to the indifference curve that goes through the point $(x_0, 0)$. Thus the merchant reaches a point like G in equilibrium.

# Problem Set 2

## 1. Product Differentiation on the Linear Road

- Marginal man:

$$V - tx - p_0 = V - t(1-x) - p_1.$$

$$x = \frac{p_1 - p_0 + t}{2t} = D_0(p_0, p_1).$$

$$1 - x = \frac{p_0 - p_1 + t}{2t} = D_1(p_0, p_1).$$

- Firm 1's problem:

$$Max\ (p_1 - c_1)(\frac{p_0 - p_1 + t}{2t}).$$

  FOC:

$$p_1 = \frac{1}{2}(p_0 + t + c_1).$$

  Symmetrically for firm 0:

$$p_0 = \frac{1}{2}(p_1 + t + c_0).$$

- Solve simultaneously:

$$p_1^* = \frac{1}{3}(3t + 2c_1 + c_0).$$

$$p_0^* = \frac{1}{3}(3t + c_1 + 2c_0).$$

- If $c_1 > c_0$, then $p_1 > p_0$ and $x > 1 - x$ so the low cost firm sets the lower price and gets the bigger market share.

- If $c_0 = c_1 = c$, then $p_0 = p_1 = t + c$.

## 2. Firms Choose Transport Costs and Prices

- In stage 1, firms choose $t_i \in [0, \infty)$. In stage 2, firms choose prices and compete Bertrand style.

- 2 firms, no costs.

- Marginal Man:

$$V - t_0x - p_0 = V - t_1(1-x) - p_1.$$

$$x = \frac{p_1 - p_0 + t_1}{t_0 + t_1} = D_0(p_0, p_1).$$

$$1 - x = \frac{p_0 - p_1 + t_0}{t_0 + t_1} = D_1(p_0, p_1).$$

- Stage 2 maximization for firm 0:

$$Max_{p_0}\ p_0 D_0.$$

FOC:

$$p_0 = \frac{t_1 + p_1}{2}.$$

Symmetrically for firm 1:

$$p_1 = \frac{t_0 + p_0}{2}.$$

- NE:

$$p_0 = \frac{1}{3}(2t_1 + t_0).$$

$$p_1 = \frac{1}{3}(2t_0 + t_1).$$

- Stage 1 problem for firm 0:

$$Max_{t_0}\ \{(\frac{2t_1 + t_0}{3})\frac{t_1 + (2/3)t_0 + (1/3)t_1 - [(2/3)t_1 + (1/3)t_0]}{t_0 + t_1}\}.$$

Simplifying:

$$Max_{t_0}\ \{\frac{[(2/3)t_1 + (1/3)t_0]^2}{t_0 + t_1}\}.$$

If $t_1 = 0$,

$$Max_{t_0}\ \{\frac{[(1/3)t_0]^2}{t_0} = \frac{t_0}{9}\}.$$

Clearly $t_0 = 0$ is NOT a best response.

- For transportation costs, $t_0 = t_1 = 0$, we have homogeneous good Bertrand competition and both firms attain zero profits. Either firm can deviate to a positive transport cost. Suppose $t_1 = 0$ and $t_0 > 0$, induces prices

$$p_0 = \frac{1}{3}(2t_1 + t_0) = \frac{t_0}{3}.$$

$$p_1 = \frac{1}{3}(2t_0 + t_1) = \frac{2t_0}{3}.$$

Marginal man:

$$x = \frac{(1/3)t_0}{t_0} = \frac{1}{3} = D_0(p_0, p_1).$$

And profits for firm 0 of:

$$\frac{t_0}{3} * \frac{1}{3} = \frac{t_0}{9} > 0.$$

- So all we've proved is that $t_0 = t_1 = 0$ is NOT a NE. The question of if a NE exists is open.

## 3. Stole Model

- Firm 0's problem:

$$Max\ E[\pi^0] = \int_0^{\overline{\theta}} (p_0(s) - \frac{1}{2}q_0(s))f(s)ds,$$

subject to IR and IC.

- Note $U^0 = u(\cdot) - p_0 = (5-\theta)q_0 - p_0$.
- Maximization becomes:

$$Max\ E[\pi^0] = \int_0^{\overline{\theta}} (5-s)q_0(s) - U^0(s) - \frac{1}{2}q_0(s)^2 ds.$$

- Integration by parts yields:

$$\int_0^{\overline{\theta}} U^0(s)f^0(s)ds = \int_0^{\overline{\theta}} [U^0(\overline{\theta}) - u_\theta(q_0(s), s, v)\frac{F(s)}{f(s)}]f(s)ds$$

- Plug into our maximization:

$$Max\ E[\pi^0] = \int_0^{\overline{\theta}} (5-s)q_0(s) - \frac{1}{2}q_0(s)^2 ds - [\int_0^{\overline{\theta}} U^0(\overline{\theta}) + q_0(s)sds.$$

$$Max\ E[\pi^0] = \int_0^{\overline{\theta}} (5-s)q_0(s) - \frac{1}{2}q_0(s)^2 - max\{U^1(1-\overline{\theta}), 0\} - q_0(s)sds.$$

FOC:

$$5 - \theta - q_0 - \theta = 0.$$

$$q_0(\theta) = 5 - 2\theta.$$

- The guy at the boundry, $\overline{\theta}$, yields no surplus to the firm so:

$$(5-2\overline{\theta})^2 - \frac{1}{2}(5-2\overline{\theta})^2 - U^{-i}(1-\overline{\theta}_i) - q^{-i}(1-\overline{\theta}_i)\overline{\theta}_i = 0.$$

By symmetry, $\overline{\theta} = \frac{1}{2}$ and if you plug in (see notes), $U^i(\overline{\theta}) = 6$.

# 8 Lecture 8: March 28, 2006

## 8.1 Vertical Control - Tirole Chap 5

- The basic setup is two firms, an upstream (M)anufacturer and a downstream (R)etailer. Both firms have market power (both monopolists) and we assume a constant marginal cost of $c$ for the M and a marginal cost of zero for R. Demand is $D(p) = q$.
- The arrangement from M to R could be of several forms:
  - (1) Linear Pricing: $T(q) = aq$, ie total compensation, $T(q)$, is just a positive constant times quantity.
  - (2) Franchise Fee: $T(q) = aq + A$. This is the simplest form of non-linear pricing.
  - (3) Resale Price Maintanence (RPM). The M restricts the price that the R can charge to end consumers.
  - (4) Quantity fixing. Similar to RPM.
- There are some problems with a couple of these. For example, (2) is a form of price discrimination which may be unavailable if arbitrage is possible. (3) may be a problem if the end price to consumers is not observable. Often times, linear pricing (1) is the only option available.
- What other types of vertical control are possible? There are three main types:
  - (1) Exclusive territory arrangements. This is when the M will restrict intrabrand competition among the retailers. This is not per se illegal.
  - (2) Exclusive dealing arrangements. This is when the M will restrict interbrand competition among the retailers. This is not per se illegal. One real example of this is a case brought by the DoJ against a false teeth manufacturer. This M made the teeth to supply denture making retailers. The M said that the denture makers could not purchase their teeth from other manufacturers. The DoJ ruled that this was “rule of reason” illegal.
  - (3) Tying arrangements. Suppose you have a retailer who uses several inputs into the production of a good. One input is supplied by a M competitively and the M has a monopoly in the other. The M says to the retailer: if you want to buy the monopolized input from me, you must also buy the competitive input. This is not per se illegal.
- The problem with a lot of these is the “Double Marginalization” which results.

**Simple Model of Double Marginalization (DM)**

- Suppose the M uses a linear price.

- Consider a vertically integrated M/R. Together they solve:

$$p^m(c) = arg\ max_p\ \{(p - c)D(p)\}.$$

And $p^m$ is the final price to consumers.

- Now consider the decentralized case where the M sells to the R at $p_w$. The retailer's problem:

$$p^*(p_w) = arg\ max_p\ \{(p - p_w)D(p)\}.$$

And the manufacturer's problem:

$$p_w^*(c) = arg\ max_x\ \{(x - c)D(p^*(p_w))\}.$$

Note that $p^m(z) = p^*(z)$ because the maximization is the same.

- Since we know $p^*$ is increasing in cost and $p_w > c$, then this implies:

$$p^*(p_w) > p^*(c).$$

So the price faced by consumers is now higher in the decentralized case compared with the joint monopoly outcome. This our DM problem.

- Note also that since $p^m(c)$ maximizes the joint profit, $p^*(p_w)$ with $p_w \neq c$ must imply the joint profit is lower.

- How can we solve the DM problem?

**Solving the DM Problem with a Franchise Fee**

- Suppose the total payment from R to M is:

$$T(q) = p_w q + A,$$

where $A$ is a lump sum payment.

- What will the M do? He will set $p_w = c$ and set $A$ equal to the R's final profits. The retailer's problem is thus:

$$p^*(p_w) = arg\ max_p\ \{(p - p_w)D(p) - A\},$$

which is the same maximization as before and if $p_w = c$, then $p^*(p_w) = p^m(c)$. So we've solved the DM problem.

- So the M extracts all the profits of the R through this mechanism.

- The problem with this is determining how profitable each franchise will be. Some McDonalds might be in better locations than others so you couldn't offer just one contract to everyone. Arbitrage must not be possible. Usually arrangements we see in practice are more complicated than this.

**Solving the DM Problem with RPM**

- Suppose the M forces the R to charge a price $p^* \underbrace{=}_{RPM} p_w$ and the M chooses: $p_w = p^m(c)$.
- The retailer again makes no profit and the DM problem is solved.
- We now turn to two complications of these models and determine if we can "solve" the DM problem in each case.

**Vertical Externalities - Downstream Moral Hazard**

- Suppose the retailer faces demand, $q = D(p, s)$, where $s$ is some level of service offered with $D_s > 0$.
- The per unit marginal cost of service is $\Phi(s)$, so the total cost of service is $q\Phi(s)$.
- Consider the integrated case. The joint M/R solve:

$$(p^m(c), s^m(c)) = arg\ max_{p,s}\ \{(p - c - \Phi(s))D(p, s)\}.$$

- In the decentralized case, the retailer solves:

$$(p^*(p_w), s^*(p_w)) = arg\ max_{p,s}\ \{(p - p_w - \Phi(s))D(p, s)\}.$$

  And again in this case, $p^m(z) = p^*(z)$ and $s^m(z) = s^*(z)$, since the maximization is the same. The manufacturer solves:

$$p_w = arg\ max_x\ \{(x - c)D(p^*(p_w), s^*(p_w))\}.$$

- So we have the same basic double marginalization problem.
- Can we solve it? We can solve it by using a Franchise fee as before though RPM won't work because the M can't influence the R's choice of service. The R will choose a suboptimal level of service.

**Vertical Externalities - Input Substitution**

- Suppose the retailer now uses two inputs, $x_1$ and $x_2$.
- Production function: $q = f(x_1, x_2)$, and suppose $p(q)$ is the inverse demand curve.
- The marginal costs for the two inputs are $c_1$ and $c_2$ respectively.
- Vertical integration case:

$$(x_1^m, x_2^m) = arg\ max_{x_1,x_2}\ \{\underbrace{p(f[x_1, x_2])}_{inv\ demand} * f(x_1, x_2) - c_1x_1 - c_2x_2\}.$$

- Now suppose $x_1$ is monopolized and $x_2$ is supplied competitively. An upstream M will choose:
$$p_{w2} = c_2, \quad p_{w1} > c_1.$$

- What will the retailer do? He will distort his use of the two inputs (probably towards $x_2$).

- Can we solve it? The franchise fee would work if $p_{w1} = c$ and the fixed fee extracts all profits. RPM would not work, though imposing a tie may work.

## 8.2 Regulation and Incentive Contracts

- This might be called "Problems in Regulation" or "The Economics of Incentives Contracts."

- The regulator faces a sequence of challenges when considering a firm or industry:
  - (1) Identify the market failure. Eg, an inefficiency, an economic justice problem, etc.
  - (2) Assess the informational asymmetry. What does the regulator know? Eg, demand, costs, etc. Can the regulator verify effort?
  - (3) Identify the potential regulatory policies. To solve the failure, the regulator can institute two types of policies:
    * (a) "Self Enforcing Policies" Force microsoft to release the code for windows.
    * (b) "Conduct Restrictions" Force microsoft to break up into an OS company and a software company.

**Loeb-Magat (1979 Journal of Law and Economics)**

- Suppose we have a (natural monopolist) firm with constant marginal cost and a regulator is coming in to do his thing.

- The regulator knows demand and fixed costs, but does NOT know the level of marginal costs.

- See G-8.1 for a diagram of the current market setting.

- The regulator proposes the following rule: "Subsidize the firm by the amount of the consumer surplus."

- Under the rule, the firm's profit when the price is $p^m$ is the area: $Ap^mB + p^mBDp^*$.

- Under the rule, the firm's profit when the price is $c$ is the area $ACp^*$ which is clearly larger.

- So the firm would have an incentive to reveal his own cost by setting the price equal to marginal cost.

- Clearly this is a damn costly regulation. The regulator could do a bit better if he at least knew that:

$$MC \in \{\underline{MC}, \overline{MC}\}.$$

In this case, consider G-8.2, the regulator could set a tax equal to the shaded region and if the firm has $MC = \overline{MC}$, he would just be indifferent between operating and shutting down and if the true $MC < \overline{MC}$, the firm would also continue to operate but would make positive profits. So this might reduce the cost of the regulation.

- So we now move to a model of Binary Uncertainty.

**Binary Uncertainty**

- Suppose the regulator wants to maximize a weighted average of consumer and producer surplus.

- Assume firm costs are governed by either $C(x; H)$ or $C(x; L)$. So the firm is either a high or low cost producer and the regulator doesn't know which (hence binary uncertainty). The regulator does know demand.

- Assume:
  - (1) $C(x; \alpha) > 0, \ \forall \, x > 0, \ \alpha \in \{H, L\}$.
  - (2) $C_x > 0$.
  - (3) $C(x; H) > C(x; L) \ \forall \, x > 0$.
  - (4) $C(x; H) - C(x'; H) > C(x; L) - C(x'; L) \ \forall \, x > x' > 0$. Ie Single Crossing Property (SCP). High cost guy is steeper than low cost guy.

- Though the regulator is uncertain about costs, he does know:

$$Pr\{H\} = \phi_H, \ \ Pr\{L\} = \phi_L.$$

- Suppose the regulator offers the firm a choice of one of two contracts:

$$\{(p_L, T_L) \ or \ (p_H, T_H)\}.$$

where $p_\alpha$ is the price the firm gets to charge its consumers and $T_\alpha$ is a lump sum that the firm must pay the regulator.

- Profits of a firm of type $\alpha$:

$$\pi(p, T|\alpha) = pD(p) - C(D(p); \alpha) - T.$$

- Clearly the regulator wants the firm to self-select, so we need the following regulatory constraints satisfied:

$$\begin{aligned} (IR_L): \quad & \pi(p_L, T_L|L) \geq \underline{\pi} \\ (IR_H): \quad & \pi(p_H, T_H|H) \geq \underline{\pi} \\ (IC_L): \quad & \pi(p_L, T_L|L) \geq \pi(p_H, T_H|L) \\ (IC_H): \quad & \pi(p_H, T_H|H) \geq \pi(p_L, T_L|H) \end{aligned}$$

  where $\underline{\pi}$ is some minimum level of profits (possibly zero) for the firm to choose to stay in the market.

- The regulator's problem is thus:

  $$Max_{(p_L,T_L,p_H,T_H)} \ \{\phi_L[\lambda(S(p_L)+T_L)+(1-\lambda)\pi(p_L,T_L|L)]+\phi_H[\lambda(S(p_H)+T_H)+(1-\lambda)\pi(p_H,T_H|H)]\},$$

  subject to:

  $$IR_L, IR_H, IC_L, IC_H.$$

  Note that $S(\cdot)$ is the level of consumer surplus at a given price and $\lambda$, the weight placed on CS and the transfer, is assumed to be greater than $\frac{1}{2}$.

- We'll solve this next week.

# 9 Lecture 9: April 4, 2006

## 9.1 More on Regulation

### More on Binary Uncertainty

- Recall the model from last time where we write the regulator's problem as:

$$Max_{(p_L,T_L,p_H,T_H)}\ \{\sum_{i\in\{L,H\}} \phi_i[\lambda(S(p_i)+T_i)+(1-\lambda)\pi(p_i,T_i|i)]\},$$

subject to:

$$\begin{array}{ll}(IR_i): & \pi(p_i,T_i|i)\geq \underline{\pi},\ i=L,H\\ (IC_i): & \pi(p_i,T_i|i)\geq \pi(p_j,T_j|i),\ (i,j)=(L,H),\ (i,j)=(H,L)\end{array}$$

- Recall the profit function is $\pi(p_i,T_i|i)=p_iD(p_i)-C(D(p_i)|i)-T_i$. If we set $\lambda=\frac{1}{2}$, meaning that we didn't care about the division between consumer surplus and profit, we would just be maximizing total surplus.

- Rewrite the objective function as follows:

$$\begin{aligned}\Phi &= \sum_{i\in\{L,H\}} \phi_i[\lambda(S(p_i)+T_i)+(1-\lambda)\pi(p_i,T_i|i)]\\ &= \sum_{i\in\{L,H\}} \phi_i[\lambda S(p_i)+\lambda T_i+(1-\lambda)(p_iD(p_i)-C(D(p_i)|i)-T_i)]\\ &= \sum_{i\in\{L,H\}} \phi_i[\lambda S(p_i)+(1-\lambda)(p_iD(p_i)-C(D(p_i)|i))+(2\lambda-1)T_i]\end{aligned}$$

So for $\lambda>\frac{1}{2}$, $2\lambda-1>0$, so $\frac{\partial\Phi}{\partial T_i}>0$. Hence we will want to make $T_i$ as large as possible to maximize $\Phi$.

- Since the low type firm has lower costs, it is clear that:

$$\pi(p_H,T_H|L)>\pi(p_H,T_H|H).$$

Add to this expession the IC constraints on either side:

$$\underbrace{\pi(p_L,T_L|L)\geq\pi(p_H,T_H|L)}_{IC_L}>\underbrace{\pi(p_H,T_H|H)\geq\pi(p_L,T_L|H)}_{IC_H}.\quad (*)$$

And now consider the first and third term of this expression:

$$\pi(p_L, T_L|L) > \underbrace{\pi(p_H, T_H|H) \geq \underline{\pi}}_{IR_H}.$$

So clearly if $IR_H$ is satisfied, then $IR_L$ will also be satisfied from this last equation. So we might as well make $IR_H$ strict (by making $T_H$ has large as possible). So:

$$\pi(p_H, T_H|H) = \underline{\pi}.$$

Or:

$$p_H D(p_H) - C(D(p_H)|H) - T_H = \underline{\pi}.$$
$$T_H^* = p_H D(p_H) - C(D(p_H)|H) - \underline{\pi}.$$

- So by setting $T_H^*$ as shown, we have taken care of the IR constraints. What about the IC's? We want to push down the low type's profits to the point where he is just indifferent between truthfully revealing and faking it (See equation (*)). That is, we want $IC_L$ to bind! Thus:

$$\pi(p_L, T_L|L) = \pi(p_H, T_H|L).$$

Which we can write:

$$p_L D(p_L) - C(D(p_L)|L) - T_L = p_H D(p_H) - C(D(p_H)|L) - T_H.$$

If we substitute $T_H^*$ into this equation, we can solve for $T_L^*$:

$$\begin{aligned}
p_L D(p_L) - C(D(p_L)|L) - T_L &= p_H D(p_H) - C(D(p_H)|L) - [p_H D(p_H) - C(D(p_H)|H) - \underline{\pi}] \\
p_L D(p_L) - C(D(p_L)|L) - T_L &= -C(D(p_H)|L) + C(D(p_H)|H) + \underline{\pi} \\
T_L &= p_L D(p_L) - C(D(p_L)|L) + C(D(p_H)|L) - C(D(p_H)|H) - \underline{\pi} \\
T_L^* &= p_L D(p_L) - C(D(p_L)|L) - [C(D(p_H)|H) - C(D(p_H)|L)] - \underline{\pi}
\end{aligned}$$

Note, we can rearrange this last equation as follows:

$$p_L D(p_L) - C(D(p_L)|L) - T_L^* = [C(D(p_H)|H) - C(D(p_H)|L)] + \underline{\pi}$$

$$\pi(p_L, T_L|L) = \underbrace{[C(D(p_H)|H) - C(D(p_H)|L)]}_{\alpha} + \underline{\pi},$$

where the $\alpha$ term is called the "Information rent" or the "Incentive payment" to the low type to produce more output (and not fake it).

- So the $IR_L$, $IR_H$, and the $IC_L$ are all satisfied. We will assume $IC_H$ holds and we'll verify it at the end.

- Rewrite our objective function:

$$\begin{aligned}\Phi &= \sum_{i\in\{L,H\}} \phi_i[\lambda S(p_i) + (1-\lambda)(p_i D(p_i) - C(D(p_i)|i)) + (2\lambda-1)T_i] \\ &= \phi_L[\lambda S(p_L) + (1-\lambda)(p_L D(p_L) - C(D(p_L)|L)) + (2\lambda-1)T_L^*] + \\ &\quad \phi_H[\lambda S(p_H) + (1-\lambda)(p_H D(p_H) - C(D(p_H)|H)) + (2\lambda-1)T_H^*]\end{aligned}$$

- Note that $S(p_L) = \int_{p_L}^{\overline{p}} D(z)dz$, so $S'(p_L) = -D(p_L)$. Take the FOC of $\Phi$ wrt $P_L$:

$$\begin{aligned}\frac{\partial \Phi}{\partial p_L} &= \phi_L[\lambda S'(p_L) + (1-\lambda)(D(p_L) + D'(p_L)[p_L - \frac{\partial C}{\partial q}]) + (2\lambda-1)\frac{\partial T_L^*}{\partial p_L}] \\ 0 &= \phi_L[-\lambda D(p_L) + (1-\lambda)(D(p_L) + D'(p_L)[p_L - \frac{\partial C}{\partial q}]) + \\ &\quad (2\lambda-1)(D(p_L) + D'(p_L)[p_L - \frac{\partial C}{\partial q}])] \\ 0 &= -\lambda D(p_L) + \lambda(D(p_L) + D'(p_L)[p_L - \frac{\partial C}{\partial q}]) \\ 0 &= \lambda(D'(p_L)[p_L - \frac{\partial C}{\partial q}]) \\ 0 &= D'(p_L)[p_L - \frac{\partial C}{\partial q}]\end{aligned}$$

So $p_L = \frac{\partial C}{\partial q}$, or price equals marginal cost !!

- So we extract all the surplus from the low type. We could do something similar for the high type which would yield:

$$p_H = \frac{\partial C}{\partial q} + \frac{\phi_L}{\phi_H}\frac{2\lambda-1}{\lambda}\left[\frac{\partial C(D(p_H)|H)}{\partial q} - \frac{\partial C(D(p_H)|L)}{\partial q}\right].$$

So the high type's price is marginal cost PLUS some positive term. So the regulator distorts the high type away from marginal cost in order to drive down the price of the low type to marginal cost. Again note that if $\lambda = \frac{1}{2}$, we don't have a distortion, but in that case, we aren't weighting consumer surplus more than firm profits.

- The last step is to verify that $IC_H$ holds. Which it does. Sweet.

**Continuous Case**

- We now move from binary uncertainty to the case there the cost uncertainty is continuous, ie:

$$C(x|c) = cx,\ c \sim F,\ f(c) > 0\ \forall\, c \in [\underline{c}, \overline{c}].$$

So the marginal cost has some density and we know the largest and smallest it can possibly be.

- Assume the pdf, $f$, is $C^1$ and the hazard rate, $\frac{F(c)}{f(c)}$, is non-decreasing in $c$.
- The regulator offers contracts:
$$\{x(c), R(c)\},$$
where $x(c)$ is the output of the firm and $R(c)$ is the resulting revenue of the firm.
- Profits of the firm are thus $R(c) - cx(c)$ and consumer surplus can be written $S(x) = \int_0^x p(z)dz$. So the regulator's probem is:
$$Max_{x,R}\left\{\int_{\underline{c}}^{\overline{c}}\{\lambda[S(x(c)) - R(c)] + (1-\lambda)[R(c) - cx(c)]\}f(c)dc\right\},$$
subject to:
$$\begin{array}{ll}(IR): & R(c) - cx(c) \geq \underline{\pi}\ \forall\ c\\ (IC): & R(c) - cx(c) \geq R(c') - cx(c')\ \forall\ c, c'\end{array}$$

- Define the truthtelling level of profits as :
$$V(c) = R(c) - cx(c),$$
ie a firm of type $c$ truthfully reveals his cost, $c$.
- The IC constraint implies:
$$V(c) \geq R(c') - cx(c').$$
Rewrite the RHS:
$$V(c) \geq R(c') - c'x(c') + c'x(c') - cx(c') = V(c') + c'x(c') - cx(c').$$
Thus,
$$V(c) - V(c') \geq [c' - c]x(c') \quad (1)$$
This holds for all $c$ and $c'$, so we can just switch the roles of $c$ and $c'$ in equation (1):
$$V(c') - V(c) \geq [c - c']x(c) \quad (2)$$
Equations (1) and (2) imply:
$$[c' - c]x(c) \geq V(c) - V(c') \geq [c' - c]x(c').$$

WLOG, assume $c > c'$, we have:

$$-[c - c']x(c) \geq V(c) - V(c') \geq -[c - c']x(c').$$

$$-x(c) \geq \frac{V(c) - V(c')}{c - c'} \geq -x(c').$$

Let $c \to c'$:

$$-x(c) \geq \frac{\partial V(c)}{\partial c} \geq -x(c).$$

So,

$$\frac{\partial V(c)}{\partial c} = -x(c).$$

- Now using this last equation and the FTC,

$$V(c) - V(c') = -\int_{c'}^{c} x(s)ds.$$

Let $c' = \overline{c}$, we have:

$$V(c) = V(\overline{c}) + \int_{c}^{\overline{c}} x(s)ds. \quad (3)$$

So the profits of a firm with marginal cost, $c$, if the IC is satisfied, must follow equation (3). That is, we have the level of profits of the highest cost firm PLUS some incentive payment or informational rents, just as we had in the binary case.

- As we had in Stole, we have the condition that "equation (3) plus $x(c)$ being non-increasing if and only if the IC holds."

- Note the producer at $\overline{c}$ is the highest cost producer and must get the lowest level of profits. So if the IR holds for him, it holds for everyone. Thus, set:

$$V(\overline{c}) = \underline{\pi}.$$

If this holds, and the IC is satisfied, then the IR holds for everyone.

- Rewrite (3) as:

$$V(c) = \underline{\pi} + \int_{c}^{\overline{c}} x(s)ds. \quad (4)$$

- Substitute (4) back into the objective function:

$$\begin{aligned}
\Phi &= \int_{\underline{c}}^{\overline{c}} \{\lambda[S(x(c)) - R(c)] + (1-\lambda)V(c)\} f(c)dc \\
&= \int_{\underline{c}}^{\overline{c}} \{\lambda[S(x(c)) - (V(c) + cx(c))] + (1-\lambda)V(c)\} f(c)dc \\
&= \int_{\underline{c}}^{\overline{c}} \{\lambda[S(x(c)) - cx(c)] - (2\lambda - 1)V(c)\} f(c)dc \\
&= \int_{\underline{c}}^{\overline{c}} \{\lambda[S(x(c)) - cx(c)] - (2\lambda - 1)[\underline{\pi} + \int_{c}^{\overline{c}} x(s)ds]\} f(c)dc
\end{aligned}$$

- Now we have to do some crazy IBP on that double integral term. So we're working with:

$$\int_{\underline{c}}^{\overline{c}} \int_{c}^{\overline{c}} x(s)ds f(c)dc.$$

Let $u = \int_{c}^{\overline{c}} x(s)ds$ and $dv = f(c)dc$. So $v = F(c)$. And $du = -x(c)dc$. So,

$$\int_{\underline{c}}^{\overline{c}} \int_{c}^{\overline{c}} x(s)ds f(c)dc = \left[ \underbrace{\int_{c}^{\overline{c}} x(s)ds}_{u} \underbrace{F(c)}_{v} \right]_{\underline{c}}^{\overline{c}} - \int_{\underline{c}}^{\overline{c}} \underbrace{-x(c)dc}_{du} \underbrace{F(c)}_{v}.$$

Or, since the first term is zero,

$$\int_{\underline{c}}^{\overline{c}} \int_{c}^{\overline{c}} x(s)ds f(c)dc = \int_{\underline{c}}^{\overline{c}} x(c)F(c)dc.$$

Which we can also write:

$$\int_{\underline{c}}^{\overline{c}} \int_{c}^{\overline{c}} x(s)ds f(c)dc = \int_{\underline{c}}^{\overline{c}} x(c)\frac{F(c)}{f(c)} f(c)dc. \quad (5)$$

- So plug (5) back into our objective function and we can then do pointwise optimization of $x(c)$.

$$\begin{aligned}
\Phi &= \int_{\underline{c}}^{\overline{c}} \{\lambda[S(x(c)) - cx(c)] - (2\lambda - 1)[\underline{\pi} + \int_{c}^{\overline{c}} x(s)ds]\} f(c)dc \\
&= \int_{\underline{c}}^{\overline{c}} \{\lambda[S(x(c)) - cx(c)] - (2\lambda - 1)[\underline{\pi} + x(c)\frac{F(c)}{f(c)}]\} f(c)dc \\
\frac{\partial \Phi}{\partial x} &= \lambda(S'(x(c)) - c) - (2\lambda - 1)\frac{F(c)}{f(c)} = 0 \\
S'(x(c)) - c &= \frac{2\lambda - 1}{\lambda}\frac{F(c)}{f(c)}
\end{aligned}$$

Note that $S'(x(c)) = p(x(c))$ (see above), so we have:

$$p(x(c)) - c = \frac{2\lambda - 1}{\lambda}\frac{F(c)}{f(c)}.$$

- Again for $c = \underline{c}$, the lowest cost producer:

$$p(x(\underline{c})) - \underline{c} = 0 \Longrightarrow p = \underline{c},$$

marginal cost pricing. But for higher cost firms, we distort prices more and more away from cost to sustain the contract.

# 10 Lecture 10: April 11, 2006

## 10.1 More on Regulation

**Correlated Signals - Kramer and McClain**

- We now consider a model where the regulator has some sort of signal about the cost structure of the monopolist which is correlated with the firm's true costs.
- If there is ANY (however small positive or negative) correlation between the signal and the cost, we can construct a mechanism to extract all the surplus and attain an efficient level of production.
- Suppose costs for the monopolist are either $c_L$ or $c_H$ and the regulator receives a signal, $s_L$ or $s_H$. Signals and costs are correlated such that:

$$Pr\{s_L|c_L\} = \psi_{LL} > Pr\{s_H|c_L\} = \psi_{HL}$$

$$Pr\{s_H|c_H\} = \psi_{HH} > Pr\{s_L|c_H\} = \psi_{LH}$$

  So the signal is just assumed to be correct more often than not.
- Model timing:
  - (1) The monopolist learns his true cost.
  - (2) The monopolist chooses one of two contracts:

$$\{x_L, (T_{LL}\ or\ T_{HL})\}\ \ OR\ \ \{x_H, (T_{HH}\ or\ T_{LH})\},$$

    where $T_{ij}$ is a tax the regulator charges when the signal is $i$ but type $j$ is announced [note that the notation is opposite from KW's presentation]. So the monopolist may choose $x_L$ say, but the tax he pays will depend on the signal that is received by the regulator. So even in a truthtelling equilibrium, there is a possibility that the signal is incorrect. The firm does NOT know the signal before choosing the contract!
  - (3) Revenues are realized:

$$R(x_L) = x_L P(x_L),\ \ or\ \ R(x_H) = x_H P(x_H).$$

- So what are the expected profits of a firm? Truthtelling firms that have high and low costs earn respectively:

$$\Pi(L|L) = x_L P(x_L) - c_L x_L - \psi_{LL} T_{LL} - \psi_{HL} T_{HL}$$

$$\Pi(H|H) = x_H P(x_H) - c_H x_H - \psi_{HH} T_{HH} - \psi_{LH} T_{LH}$$

  So for instance the last term in the second expression is the tax paid by the high cost firm when the signal is (incorrectly) low and the announcement is high, weighted by the probability of that happening.

- So when designing contracts that induce truthful reporting, we need some IC and IR constraints to hold. Let $\pi(i|j) = R(x_i) - c_j x_i$. We need:

$$
\begin{aligned}
(IR_L): &\quad \pi(L|L) - \psi_{LL}T_{LL} - \psi_{HL}T_{HL} \geq \underline{\pi} \\
(IR_H): &\quad \pi(H|H) - \psi_{HH}T_{HH} - \psi_{LH}T_{LH} \geq \underline{\pi} \\
(IC_L): &\quad \pi(L|L) - \psi_{LL}T_{LL} - \psi_{HL}T_{HL} \geq \pi(H|L) - \psi_{LL}T_{LH} - \psi_{HL}T_{HH} \\
(IC_H): &\quad \pi(H|H) - \psi_{HH}T_{HH} - \psi_{LH}T_{LH} \geq \pi(L|H) - \psi_{LH}T_{LL} - \psi_{HH}T_{HL}
\end{aligned}
$$

- So we claim that there exists an incentive contract that satisfies all these constraints with the IR and IC constraints holding with equality! So we extract all surplus and we don't have a distortion like we had in the last models. If $x_L^*$ and $x_H^*$ are the efficient output levels for $c_L$ and $c_H$, we only have 4 unknowns (the 4 tax levels). So we want to solve the 4 equations above (IRs and ICs) to find the tax levels.

- Write the problem in matrix form:

$$
\begin{bmatrix} \pi(L|L) - \underline{\pi} \\ \pi(H|H) - \underline{\pi} \\ \pi(L|L) - \pi(H|L) \\ \pi(H|H) - \pi(L|H) \end{bmatrix} = \begin{bmatrix} \psi_{LL} & 0 & \psi_{HL} & 0 \\ 0 & \psi_{LH} & 0 & \psi_{HH} \\ \psi_{LL} & -\psi_{LL} & \psi_{HL} & -\psi_{HL} \\ -\psi_{LH} & \psi_{LH} & -\psi_{HH} & \psi_{HH} \end{bmatrix} * \begin{bmatrix} T_{LL} \\ T_{LH} \\ T_{HL} \\ T_{HH} \end{bmatrix}.
$$

- As long as that matrix of $\psi$'s is of full rank (non-singular / nonzero determinant), then there exists a unique solution to the problem. This means that there is a solution if:

$$\psi_{LL}\psi_{HH} \neq \psi_{LH}\psi_{HL},$$

  or if the signal is NOT independent of the cost level. Note if all the $\psi$'s equalled $\frac{1}{2}$, this condition would fail, the signal would be meaningless and we couldn't find a solution.

- So with some sort of corrleation, we will have a solution and all IC and IR constraints will bind (in expectation).

- The optimal contract will be one in which $T_{HH}$ and $T_{LL}$ are huge and negative (ie, we reward truthtelling), and $T_{HL}$ and $T_{LH}$ are huge and positive (ie, we shoot the liars in the head).

- In equilibrium, the firm is always truthfully revealing, but we still shoot him in the head from time to time if the signal comes up wrong.

- This may not sound realistic because the IR and IC constraints only hold ex-ante in expectation (and firms generally don't like getting shot in the head). Ie, ex-post, the shoot him in the head scenario will sometimes come up and when it does, the IR's will not hold. However, it can be shown that if the correlation between the signal and the costs is relatively high, the skewness of the taxes that we require is smaller so we don't have to punish the liars quite as much to sustain the equilibrium.

**Repeated Interaction (Ratchet Effect) - Lafont and Tirole**

- The last topic in the regulation section will consider a model of a regulator and monopolist interacting in two periods sequentially. We assume the regulator CANNOT commit to a contract in the second period and this mean a truthtelling equilibrium will not exist. In period 1, the monopolist will inflate his announced cost above his true cost.
- The regulator is assumed to know that $c \in [\underline{c}, \overline{c}]$ with density $f(c)$.
- The contract offered in the first period is:

$$\{x(c), R(c)\},$$

  where the firm announces $c$, gets to produce $x(c)$, and receives revenues in return of $R(c)$.
- **Claim** If the first period contract induces a separating equilibrium, that is $x(\cdot)$ is strictly decreasing in the announced cost, then the regulator in the second period requires the efficient level of output and will extract all surplus. In effect, the first period contract reveals the true cost of the monopolist exactly so in the second period, the regulator will just force the efficient outcome.
- We will show that if the regulator cannot commit to NOT extract all surplus in period 2, then no separating equilibrium will exist in the first period.
- Three things can happen:
  - If in period 1, the firm announced his true cost, the regulator in period 2 would set:

$$R^{(2)}(c) - cx^{(2)}(c) = \underline{\pi},$$

    ie, the regulator would force the period 2 IR constraint to bind because he knows the firm's true cost, $c$. The regulator extracts all surplus.
  - Alternatively, if the firm announced an reduced cost in period 1, $c' < c$, then:

$$R^{(2)}(c') - cx^{(2)}(c') < \underline{\pi},$$

    ie, the firm will not produce in period 2 because his IR constraint is violated.
  - Finally, if the firm announced an inflated cost in period 1, $c' > c$, then:

$$R^{(2)}(c') - cx^{(2)}(c') > \underline{\pi},$$

    ie, the firm produces in period 2 and the regulator fails to extract all surplus since the IR constraint is slack.
- Denote the truthtelling level of profits as:

$$V(c) = R(c) - cx(c) + R^{(2)}(c) - cx^{(2)}(c) = R(c) - cx(c) + \underline{\pi}.$$

- Assume $c' > c$. The following IC constraint must hold:

$$\begin{aligned}(IC):\ V(c) &\geq R(c') - cx(c') + \underbrace{R^{(2)}(c')}_{\underline{\pi}+c'x^{(2)}(c')} - cx^{(2)}(c') \\ &= \underbrace{R(c') + \underline{\pi}}_{V(c')+c'x(c')} - cx(c') + c'x^{(2)}(c') - cx^{(2)}(c') \\ &= V(c') + c'x(c') - cx(c') + c'x^{(2)}(c') - cx^{(2)}(c') \\ V(c) - V(c') &\geq [c' - c][x(c') + x^{(2)}(c')] \quad (1)\end{aligned}$$

- Assume $c' < c$ so the firm does not produce in period 2. The following IC constraint must hold:

$$\begin{aligned}(IC):\ V(c) &\geq R(c') - cx(c') + \underline{\pi} \\ &= V(c') + c'x(c) - cx(c') \\ V(c) - V(c') &\geq [c' - c]x(c') \\ &\quad \text{switch roles of } c \text{ and } c' \\ V(c') - V(c) &\geq [c - c']x(c) \\ V(c) - V(c') &\leq [c' - c]x(c) \quad (2)\end{aligned}$$

- Combine (1) and (2):

$$[c' - c]x(c) \geq V(c) - V(c') \geq [c' - c][x(c') + x^{(2)}(c')]$$

$$-x(c) \leq \frac{V(c) - V(c')}{c - c'} \leq -[x(c') + x^{(2)}(c')]$$

  For $c$ close to $c'$, this cannot hold. No fully revealing equilibrium exists in the first period because the IC will not be satisfied!

- The problem is the regulator cannot commit to NOT extract all surplus in the second period. This generates a strange incentive in period 1. There is a kink in the second period profit function which causes an incentive to not truthfully reveal your cost in the first period.

- The term "ratchet effect" refers to the fact that by truthfully revealing your cost in period 1, the regulator can ratchet the firm into a surplus extracting contract in period 2.

# 11 Lecture 11: April 18, 2006

## 11.1 Theory of the Firm

**Preliminary - What is a Firm?**

- We can consider the definition of a firm in several different ways.
  - (1) Economic definition. In terms of the models we have been studying, how do we define a firm? We might say a firm is an entity that takes resources and provides an output through a production function. However, how we define a production function is ambiguous. We also might say that a firm is an entity that can be treated as a single decision maker. In reality, there may be multiple profit centers within a firm, each acting relatively independently.
  - (2) Definition based on economic predictions regarding transactions. Some transactions take place inside a firm and some take place outside. An inside transaction might be something that takes place between two sectors of a firm which are working together to produce a product. Every transaction (inside or outside) is governed by a contract and the ones inside the firm place control in the firm's hands. The ones outside the firm (at arm's length) may necessitate complicated contracts to properly align the incentives of the firm and the outside party.
  - (3) Legal definition. Is a firm controlled by a single decision maker?
    - If yes: price fixing is ok.
    - If no, then ask: Are the divisions competitors or potential competitors?
      - If no: price fixing is ok.
      - If yes, then by a rule of reason arg, price fixing may violate antitrust laws.
- For the remainder of this section, we focus on issues involving number 2.

**Incomplete Contracts and Transaction Costs**

- There are 4 types of transaction costs that may lead to incomplete contracts:
  - (1) Some contingencies may not be foreseeable at the time of contracting.
  - (2) Even if foreseeable, there can be too many to specify.
  - (3) Monitoring compliance is costly.
  - (4) Enforcing the contract is costly.

  So given these, arm's length contracts may be costly (or infeasible), so it might be cheaper for the firm to internalize the transaction.
- For example, when Microsoft brought out Windows, they needed mice attached to computers. Instead of contracting with IBM to make and sell mice, they decided to make the mice themselves. Maybe it's mouses?

**Dynamic Contracting or Bargaining**

- Consider a two period model, $t = 1$ and $t = 2$.
- The buyer has value $V$ and the seller has cost $c$. There are gains from trade if $V > c$.
- If $V$ and $c$ are common knowledge, then bargaining should result in efficient trade.
- However, suppose there is asymmetric information. We might get an inefficiency and the seller may want to (somehow) internalize the transaction.
- Now suppose the seller has commitment power. The buyer privately knows $V$, but the sellers cost is public. Given $V \sim F$, the seller maximizes:

$$\pi_{seller} = (p - c)[1 - F(p)].$$

FOC:

$$1 - F(p) - (p - c)f(p) = 0 \Rightarrow 1 - F(p^*) = (p^* - c)f(p^*).$$

If, $F(p^*) > F(c)$, then there are some buyers with $V > c$ that do NOT buy. This is our inefficiency. See G-11.1.

- So what kind of contract should the seller offer in $t = 1$ if the buyer is not immediately aware of his value? Consider $p = c$ plus a side payment equal to $\psi$, such that:

$$\psi = E[\pi_{buyer}] = \int_{p=c}^{\infty} (V - p)f(v)dv.$$

So $\psi$ is the expected surplus to the buyer given a price of $p = c$. It's the most he's willing to pay. The seller extracts all surplus from the buyer (ex-ante). In this situation, the buyer gets to decide to accept the seller's offer. Since he has the private information, the buyer has the power. Not sure if the timing is correct here. When do the parties know $V$ ?

**Hold-up Problem**

- In this section, we suppose that some party to a transaction can make a "transaction specific investment" that will lead to a hold-up problem.
- Discrete example. Suppose the buyer's value is known at $V = 3$. The seller can make an investment to alter his cost:

$$c = \begin{cases} 0 & \text{if } I = 2 \\ 4 & \text{if } I = 0 \end{cases}$$

So if $I = 0$, then $c = 4 > 3 = V$, so no trade occurs.

- What if the buyer and seller bargained ex-post and split the surplus. If $I = 2$, then the surplus would be $V - c = 3 - 0 = 3$. Split the surplus means $p = \frac{3}{2}$. But then the seller gets $p - I = \frac{3}{2} - 2 < 0$. So the seller would not invest. This is our hold-up problem.

- More involved example. Let $p(V, c)$ be the bargained price. Ie, given a buyer's value, $V$, and seller's cost, $c$, they reach a price $p$. Let $c(I)$ be the cost of investment $I$. The seller's problem:

$$Max_I \, \{[p(V, c(I)) - c(I)] * \mathbb{1}_{\{\underbrace{V \geq c(I)}_{trade\ happens}\}} - I\}.$$

  For $I$ such that $V \geq c(I)$, our FOC is:

$$\left[\frac{dp}{dc} - 1\right] c'(I^*) - 1 = 0.$$

- The first best solution meanwhile is the solution to the social planner's problem:

$$Max_I \, \{V - c(I) - I\}.$$

  FOC:

$$c'(I^*) = -1.$$

  Note $c'(I^*)$ is the marginal effect of investment on cost.

- The $I^*$'s are the same if $\frac{dp}{dc} = 0$, ie the seller captures all the effects of investment. In general this will not be true so $I^*$ will not reach the efficient (SP) level and we have again the hold-up problem. We get under-investment by the firm since the buyer also benefits from some of the investment. Again, the firm may want to internalize this.

# Problem Set 3

## 1. Quantity Forcing

- Integrated problem:

  $$Max_{p,s}\ [p - c - \Phi(s)]D(p,s).$$

  FOC:

  $$\frac{\partial \pi}{\partial p} = [p - c - \Phi(s))\frac{\partial D}{\partial p} + D(p,s) = 0.$$

  $$\frac{\partial \pi}{\partial s} = [p - c - \Phi(s))\frac{\partial D}{\partial s} - D(p,s)\Phi' = 0.$$

  This implies:

  $$\frac{-D(p,S)}{\Phi' D(p,s)} = \frac{D_p}{D_s} \Rightarrow \Phi'(s) = -\frac{D_p}{D_s}.$$

- Unintegrated problem. The retailer solves:

  $$Max_{p,s}\ [p - p_w - \Phi(s)]D(p,s),$$

  subject to:

  $$D(p^m, s^m) = q.$$

  Lagrangian:

  $$\mathcal{L} = [p - p_w - \Phi(s)]D(p,s) - \lambda[q - D(p,s)].$$

  FOCs:

  $$\frac{\partial \mathcal{L}}{\partial p} = D(p,s) + [p - p_w - \Phi(s) + \lambda]\frac{\partial D}{\partial p} = 0.$$

  $$\frac{\partial \mathcal{L}}{\partial s} = -D(p,s)\Phi' + [p - p_w - \Phi(s) + \lambda]\frac{\partial D}{\partial s} = 0.$$

  $$\frac{\partial \mathcal{L}}{\partial lambda} \Rightarrow q = D(p^m, s^m).$$

  First two partials imply:

  $$\frac{-D(p,s)}{D(p,s)\Phi'} = \frac{D_p}{D_s} \Rightarrow \Phi'(s) = -\frac{D_p}{D_s},$$

  which is the same condition as above. Thus, given quantity forcing, the same $p$ and $s$ are chosen. All profits are extracted.

## 2a. Binary Uncertainty

- When is it optimal for the regulator to have the high cost firm produce no output (ie, his IR constraint is violated) and only the low cost firm produce?

- Recall the regulator's problem:

$$Max_{(p_L,T_L,p_H,T_H)} \{ \sum_{i\in\{L,H\}} \phi_i[\lambda(S(p_i)+T_i)+(1-\lambda)\pi(p_i,T_i|i)]\},$$

subject to:

$$\begin{aligned} (IR_i): &\quad \pi(p_i,T_i|i) \geq \underline{\pi},\ i=L,H \\ (IC_i): &\quad \pi(p_i,T_i|i) \geq \pi(p_j,T_j|i),\ (i,j)=(L,H),\ (i,j)=(H,L) \end{aligned}$$

- If only the low cost firm produces, then only $IR_L$ needs to be satisfied and we can get the efficient level of output from the firm:

$$p_L = c'(D(p_L)|L),$$

$$T_L = p_L D(p_L) - c(D(p_L)|L) - \underline{\pi}.$$

- Regulator's payoff becomes:

$$\begin{aligned} \Phi &= \sum_{i\in\{L,H\}} \phi_i[\lambda(S(p_i)+T_i)+(1-\lambda)\pi(p_i,T_i|i)] \\ &= \phi_L[\lambda(S(p_L)+T_L)+(1-\lambda)\pi(p_L,T_L|L)] \\ &= \phi_L[\lambda(S(p_L)+p_L D(p_L)-c(D(p_L)|L)-\underline{\pi})+(1-\lambda)[p_L D(p_L)-c(D(p_L)|L)-T_L]] \\ &= \phi_L[\lambda(S(p_L)-\underline{\pi})+(1-\lambda)\underline{\pi}] \\ &= \phi_L[\lambda S(p_L)+(1-2\lambda)\underline{\pi}] \end{aligned}$$

A bit different than what KW has. Seems right though?

- So if this $\Phi$ is greater than the payoffs when they both produce, then the regulator should only have the low cost firm produce.

## 2b. Regulator can Purchase Signal

- Suppose the regulator imposes $p_L = c'(D(p_L)|L)$ if the low cost is announced. If the high cost is announced, then audit with probability $s$ and tax $T_{HH}$ or $T_{HL}$ depending on the outcome out of the audit. If (with probability $1-s$), he does not audit, then tax at $T_{HH}$.

- Our IC constraints become:

$$(IC_L):\ \ \pi(x_L|L)-T_L \geq \pi(x_H|L) - \underbrace{(1-s)T_{HH}}_{no\ audit} - \underbrace{s[\psi_{LL}T_{HL}+\psi_{HL}T_{HH}]}_{audit}. \quad (1)$$

$$(IC_H): \quad \pi(x_H|H) - \underbrace{(1-s)T_{HH}}_{no\ audit} - \underbrace{s[\psi_{LH}T_{HL} + \psi_{HH}T_{HH}]}_{audit} \geq \pi(x_L|H) - T_L. \ (2)$$

- Our IR constraints become:

$$(IR_L): \quad \pi(x_L|L) - T_L \geq \underline{\pi}. \ (3)$$

$$(IR_H): \quad \pi(x_H|H) - \underbrace{(1-s)T_{HH}}_{no\ audit} - \underbrace{s[\psi_{LH}T_{HL} + \psi_{HH}T_{HH}]}_{audit} \geq \underline{\pi}. \ (4)$$

- Since we extract all surplus from the low type, equation 3 binds. If $IR_H$ is satisfied, so is $IC_H$. Since (4) implies (2) and (1) implies (3), we just need a $T_{HL}$ and $T_{HH}$ that satisfy (1) and (4) to get a solution.

- I'm not convinced.

## 3. Demand Uncertainty

- Inverse demand is $p = (\theta - x)$ where the regulator knows: $\theta \in [\underline{\theta}, \overline{\theta}]$ with density $f$. $\theta$ is known only to the monopolist.

- Costs: $c(x) = 0.5x^2$. Contracts: $(x(\theta), T(\theta))$ where $T$ is a tax given announced $\theta$.

- Monopolist's problem

$$Max\ \{\pi(x|\theta) = [\theta - x(\theta)]x(\theta) - 0.5x(\theta)^2 - T(\theta)\}.$$

- Regulator's problem:

$$Max_{x,T}\ \Phi = \left\{ \int_{\underline{\theta}}^{\overline{\theta}} \{\lambda(S(x(\theta)) + T(\theta)) + (1-\lambda)(\pi(x|\theta))\} f(\theta) d\theta \right\},$$

subject to:

$$(IR): \quad \pi(x|\theta) \geq \underline{\pi}$$

$$(IC): \quad \pi(x|\theta) \geq \pi(x|\theta')\ \forall\ \theta, \theta'$$

- Let's work with the monopolist for a bit. IC implies:

$$\begin{aligned}
\pi(\theta) &\geq \pi(\theta') \\
&= [\theta - x(\theta')]x(\theta') - 0.5x(\theta')^2 - T(\theta') \\
&= [\theta - x(\theta')]x(\theta') - 0.5x(\theta')^2 - T(\theta') + \theta' x(\theta') - \theta' x(\theta') \\
&= \pi(\theta') - \theta' x(\theta') + \theta x(\theta') \\
\pi(\theta) - \pi(\theta') &\geq x(\theta')[\theta - \theta']
\end{aligned}$$

Also, switching the roles:

$$\begin{aligned}\pi(\theta') - \pi(\theta) &\geq x(\theta)[\theta' - \theta] \\ \pi(\theta) - \pi(\theta') &\leq x(\theta)[\theta - \theta']\end{aligned}$$

Which means:

$$x(\theta')[\theta - \theta'] \leq \pi(\theta) - \pi(\theta') \leq x(\theta)[\theta - \theta'].$$

$$x(\theta') \leq \frac{\pi(\theta) - \pi(\theta')}{\theta - \theta'} \leq x(\theta).$$

Let $\theta \to \theta'$:

$$\pi'(\theta) = x(\theta).$$

By the FTC:

$$\pi(\theta) - \pi(\theta') = \int_{\theta'}^{\theta} x(s)ds.$$

$$\pi(\theta) = \pi(\theta') + \int_{\theta'}^{\theta} x(s)ds.$$

At $\theta' = \underline{\theta}$, noting that $\pi(\underline{\theta}) = \underline{\pi} = 0$ (normalized):

$$\pi(\theta) = \pi(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} x(s)ds.$$

$$\pi(\theta) = \int_{\underline{\theta}}^{\theta} x(s)ds.$$

- So back to our regulator's problem, noting that consumer surplus is: $S(x(\theta)) = \int_0^{x(\theta)}(\theta - s)ds - (\theta - x(\theta))x(\theta)$:

$$\begin{aligned}\Phi &= \int_{\underline{\theta}}^{\overline{\theta}} \{\lambda(S(x(\theta)) + T(\theta)) + (1-\lambda)(\pi(x|\theta))\} f(\theta)d\theta \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \{\lambda(S(x(\theta)) + [\theta - x(\theta)]x(\theta) - 0.5x(\theta)^2 - \pi(x|\theta)) + (1-\lambda)(\pi(x|\theta))\} f(\theta)d\theta \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \{\lambda(\int_0^{x(\theta)} (\theta - s)ds - (\theta - x(\theta))x(\theta) + [\theta - x(\theta)]x(\theta) - 0.5x(\theta)^2) + \\ &\quad (1-2\lambda)(\pi(x|\theta))\} f(\theta)d\theta \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \{\lambda(\int_0^{x(\theta)} (\theta - s)ds - 0.5x(\theta)^2) + (1-2\lambda)(\pi(x|\theta))\} f(\theta)d\theta \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \{\lambda(\int_0^{x(\theta)} (\theta - s)ds - 0.5x(\theta)^2) + (1-2\lambda)(\int_{\underline{\theta}}^{\theta} x(s)ds)\} f(\theta)d\theta\end{aligned}$$

- IBP Step, noting that $F(\overline{\theta}) = 1$ and $F(\underline{\theta}) = 0$:

$$\begin{aligned}
Q &= \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} x(s) ds f(\theta) d\theta \\
&= \left[ \int_{\underline{\theta}}^{\theta} x(s) ds F(\theta) \right]_{\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} x(\theta) F(\theta) d\theta \\
&= 1 * \int_{\underline{\theta}}^{\overline{\theta}} x(s) ds - \int_{\underline{\theta}}^{\overline{\theta}} x(\theta) F(\theta) d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} (1 - F(\theta)) x(\theta) d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \frac{(1 - F(\theta))}{f(\theta)} x(\theta) f(\theta) d\theta
\end{aligned}$$

- Plug $Q$ back in:

$$\Phi = \int_{\underline{\theta}}^{\overline{\theta}} \{\lambda( \int_0^{x(\theta)} (\theta - s) ds - 0.5 x(\theta)^2) + (1 - 2\lambda)( \frac{(1 - F(\theta))}{f(\theta)} x(\theta))\} f(\theta) d\theta$$

- Now do pointwise optimization:

$$\frac{\partial \Phi}{\partial x} = \lambda[(\theta - x(\theta)) - x(\theta)] + (1 - 2\lambda) \frac{(1 - F(\theta))}{f(\theta)} = 0.$$

Or,

$$\begin{aligned}
\lambda[(\theta - x(\theta)) - x(\theta)] &= (2\lambda - 1) \frac{(1 - F(\theta))}{f(\theta)} \\
\theta - 2x(\theta) &= \frac{(2\lambda - 1)}{\lambda} \frac{(1 - F(\theta))}{f(\theta)} \\
x(\theta) &= \frac{\theta}{2} \underbrace{- \frac{(2\lambda - 1)}{2\lambda} \frac{(1 - F(\theta))}{f(\theta)}}_{distortion!}
\end{aligned}$$

Since when output is efficient, $x(\theta) = \frac{\theta}{2}$, we have a distortion. For the high type, ie, the type with the best draw of $\theta$,

$$x(\overline{\theta}) = \frac{\overline{\theta}}{2},$$

efficient! But for everyone else, output is not efficient.