MEI
Exam Review

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1 Final Exam Revision Notes

1.1 Random Rules and Formulas

- Linear transformations of random variables.

$$f_y(Y) = f_x(X) \cdot \left|\frac{dx}{dg}\right|.$$

- Inverse Proof.

$$(AB)(AB)^{-1} = I.$$
$$(B^{-1}A^{-1})(AB)(AB)^{-1} = (B^{-1}A^{-1}).$$
$$(AB)^{-1} = (B^{-1}A^{-1}).$$

- Simple linear regression coefficient with only one independent variable and an intercept.

$$\hat{\beta} = (X'X)^{-1}X'y = \begin{bmatrix} T & \sum x_t \\ \sum x_t & \sum x_t^2 \end{bmatrix} \cdot \begin{bmatrix} \sum y_t \\ \sum x_t y_t \end{bmatrix}. \tag{1}$$

- Proof about projections.

$$\begin{aligned}
\hat{\epsilon} &= M_X \epsilon \\
&= (I - X(X'X)^{-1}X')(y - X\beta) \\
&= y - X(X'X)^{-1}X'y - X\beta + X(X'X)^{-1}X'X\beta \\
&= y - X(X'X)^{-1}X'y \\
&= M_X y \\
&= (I - X(X'X)^{-1}X')y \\
&= y - X(X'X)^{-1}X'y \\
&= y - X\hat{\beta} \\
&= \hat{\epsilon}.
\end{aligned}$$

- Simple linear regression with a constant.

$$y = \beta_1 + \beta_2 x_t + \epsilon_t.$$

$$\hat{\beta}_2 = \frac{\sum_t (x_t - \bar{x}) y_t}{\sum_t (x_t - \bar{x})^2}.$$

- R squared.

$$R^2 = 1 - \frac{RSS}{TSS} = \frac{ESS}{TSS}.$$

- Inverse Rule.

$$A^{-1} = \frac{A^*}{|A|}.$$

If A is a $2x2$, “Switch the Main, Negate the Off.”

- F test of a restriction.

$$F = \frac{(RSS_R - RSS_U)/q}{RSS_U/(N-K)} \sim F(q, N-K).$$

With q restrictions and K parameters.

- Just algebra.

$$X'X = \sum_1^T x_t x_t'.$$

$$X'\epsilon = \sum_1^T x_t \epsilon_t.$$

- Variance of the OLS estimator.

$$\hat{Var}(\hat{\beta}) = s^2 (X'X)^{-1},$$

with,

$$s^2 = \frac{RSS}{N-K} = \frac{\hat{\epsilon}'\hat{\epsilon}}{N-K} = \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{N-K}.$$

Multiplying out RSS,

$$\begin{aligned}
RSS &= (y - X\hat{\beta})'(y - X\hat{\beta}) \\
&= (y' - \hat{\beta}'X')(y - X\hat{\beta}) \\
&= y'y - \hat{\beta}'X'y - y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}) \\
&= y'y - \hat{\beta}'X'y - y'X(X'X)^{-1}X'y + ((X'X)^{-1}X'y)'X'X(X'X)^{-1}X'y \\
&= y'y - \hat{\beta}'X'y - y'X(X'X)^{-1}X'y + y'X\underbrace{(X'X)^{-1}X'X}_{I}(X'X)^{-1}X'y \\
&= y'y - \hat{\beta}'X'y \underbrace{- y'X(X'X)^{-1}X'y + y'X(X'X)^{-1}X'y}_{0} \\
&= y'y - \hat{\beta}'X'y. \\
&= y'y - y'X\hat{\beta}.
\end{aligned}$$

Thus,

$$s^2 = \frac{RSS}{N-K} = \frac{y'y - \hat{\beta}'X'y}{N-K}.$$

- Mean Lag.

$$\delta(L) = \sum_{j=0}^{\infty} \delta_j L^j.$$

(A general lag polynomial.)

$$\delta'(L) = \sum_{j=0}^{\infty} j\delta_j L^{j-1}.$$

Then,

$$Mean\ Lag \equiv \frac{\delta'(1)}{\delta(1)}.$$

- Lagged dependents PLUS $AR(1)$ errors yields inconsistent estimators!
- Autoregressive Final Form.

$$\begin{aligned} A(L)y &= B(L)X + \epsilon. \\ y &= A^{-1}BX + A^{-1}\epsilon. \\ y &= \frac{A^*}{|A|}BX + \frac{A^*}{|A|}\epsilon. \\ |A|y &= A^*BX + A^*\epsilon. \end{aligned}$$

1.2 Characteristic Polynomials

- Consider an $AR(2)$ process:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t.$$

- Rewriting with the lag operator:

$$y_t - \phi_1 L y_t - \phi_2 L^2 y_t = \epsilon_t.$$
$$(1 - \phi_1 L - \phi_2 L^2) y_t = \epsilon_t.$$

- Factorizing the left hand side:

$$1 - \phi_1 L - \phi_2 L^2 = (1 - Z_1 L)(1 - Z_2 L).$$

- Divide through by L^2:

$$L^{-2} - \phi_1 L^{-1} - \phi_2 = L^{-2}(1 - Z_1 L)(1 - Z_2 L).$$

- Multiply out the right hand side:

$$L^{-2} - \phi_1 L^{-1} - \phi_2 = L^{-2}(1 - Z_2 L - Z_1 L + Z_1 Z_2 L^2).$$
$$L^{-2} - \phi_1 L^{-1} - \phi_2 = L^{-2} - Z_2 L^{-1} - Z_1 L^{-1} + Z_1 Z_2.$$
$$L^{-2} - \phi_1 L^{-1} - \phi_2 = L^{-2} - (Z_1 + Z_2) L^{-1} + Z_1 Z_2.$$
$$L^{-2} - \phi_1 L^{-1} - \phi_2 = (L^{-1} - Z_1)(L^{-1} - Z_2).$$

- Let $Z = L^{-1}$:

$$Z^2 - \phi_1 Z - \phi_2 = (Z - Z_1)(Z - Z_2).$$

- Thus the roots of the characteristic polynomial are Z_1 and Z_2. And explicitly:

$$Z_{1,2} = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}.$$

1.3 Instrumental Variables

- Consider a regression equation:

$$y = X\beta + \epsilon.$$

If $E[X\epsilon] \neq 0$, then use Instrumental Variables (IV).

- Suppose there exists a set of instruments Z, for the problem X variable. We will use the 2 stage least squares technique to derive the IV estimator.

- Step 1: Run the following regression:

$$X = Z\gamma + u.$$

The estimator will be the following form:

$$\hat{\gamma} = (Z'Z)^{-1}Z'X.$$

- Step 2: Compute the fitted values such that:

$$\hat{X} = Z\hat{\gamma} = Z(Z'Z)^{-1}Z'X.$$

- Step 3: Run the following regression:

$$y = \hat{X}\beta + \epsilon.$$

The estimator will be the following:

$$\hat{\beta} = (\hat{X}'\hat{X})^{-1}\hat{X}'y.$$

- Substitute in for $\hat{X}$:

$$\hat{\beta} = ((Z(Z'Z)^{-1}Z'X)'Z(Z'Z)^{-1}Z'X)^{-1}(Z(Z'Z)^{-1}Z'X)'y.$$

Simplifying,

$$\hat{\beta} = (X'Z(Z'Z)^{-1}Z'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y.$$

$$\hat{\beta}_{IV} = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y.$$

Which is the IV estimator.

- An important note. If X is exactly identified in that Z is the same dimension as X, then we can carry the analysis further. Inverting:

$$\hat{\beta}_{IV} = (Z'X)^{-1}(Z'Z)\underbrace{(X'Z)^{-1}X'Z}_{I}(Z'Z)^{-1}Z'y.$$

$$\hat{\beta}_{IV} = (Z'X)^{-1}\underbrace{(Z'Z)(Z'Z)^{-1}}_{I}Z'y.$$

$$\hat{\beta}^{*}_{IV} = (Z'X)^{-1}Z'y.$$

1.4 Dicky Fuller Tests

- Use DF tables for tests of stationarity with:

$$H_0 : \phi = 1, \quad Non - Stationary.$$

$$H_1 : \phi < 1, \quad Stationary.$$

- A Guide to Dicky Fuller Tables.

$$\left\{ \begin{array}{l|l|l} \text{First Panel} & \text{Nothing} & y = \phi y_{t-1} + \epsilon_t \\ \text{Second Panel} & \text{Constant} & y = \alpha + \phi y_{t-1} + \epsilon_t \\ \text{Third Panel} & \text{Constant and Time Trend} & y = \alpha + \beta t + \phi y_{t-1} + \epsilon_t \end{array} \right. \tag{2}$$

- Example. Consider the model:

$$y_t = \alpha + \phi y_{t-1} + \epsilon_t.$$

To test:

$$H_0 : \phi = 1, \alpha = 0, \quad (random\ walk).$$

$$H_1 : \phi < 1, \alpha <=> 0.$$

Use:

$$\tau_u = \frac{\hat{\phi} - 1}{SE(\hat{\phi})}.$$

And check against middle panel.

If test is anything other than $H_0 : \phi = 1$, **use t test!**

1.5 Lagrange Multiplier Test

- We'll use the LM test to test for serially correlated errors. Consider the model:

$$y_t = x_t'\beta + u_t, \quad u_t = \phi u_{t-1} + \epsilon_t, \quad |\phi| < 1, \quad \epsilon_t \sim\ iid.$$

- Test:

$$H_0 : \phi = 0 \text{ (Errors NOT Serially Correlated).}$$

$$H_1 : \phi \neq 0 \text{ (Errors Serially Correlated).}$$

- Rewrite the model by lagging once, multiplying by ϕ and subtracting,

$$y_t - \phi y_{t-1} = x_t'\beta - \phi x_{t-1}'\beta + u_t - \phi u_{t-1}.$$

$$y_t = \phi y_{t-1} + x_t'\beta - \phi x_{t-1}'\beta + \epsilon_t.$$

- Log Likelihood:

$$Ln(L) = -\frac{(T-1)}{2}ln(2\pi) - \frac{(T-1)}{2}ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=2}^{T}\epsilon_t^2.$$

With,

$$\epsilon_t = y_t - \phi y_{t-1} - x_t'\beta + \phi x_{t-1}'\beta.$$

- First Order Conditions.

$$\frac{\partial ln(L)}{\partial \phi} = \frac{1}{\sigma^2}\sum_{t=2}^{T}\epsilon_t(-y_{t-1} + x_{t-1}'\beta).$$

$$\frac{\partial ln(L)}{\partial \beta} = \frac{1}{\sigma^2}\sum_{t=2}^{T}\epsilon_t(-x_t' + \phi x_{t-1}').$$

- To simplify, let:

$$z_t = -\frac{\partial \epsilon_t}{\partial \psi}.$$

With $\psi = (\phi\ \beta)'$.

- Thus,

$$z_t = -\begin{bmatrix} \frac{\partial \epsilon_t}{\partial \phi} \\ \frac{\partial \epsilon_t}{\partial \beta} \end{bmatrix} = -\begin{bmatrix} -y_{t-1} + x_{t-1}'\beta \\ -x_t' + \phi x_{t-1}' \end{bmatrix} = \begin{bmatrix} y_{t-1} - x_{t-1}'\beta \\ x_t' - \phi x_{t-1}' \end{bmatrix}. \quad (3)$$

- So the first order conditions simplify to:

$$\sum_{t=2}^{T} z_t\epsilon_t = 0.$$

- So the next step is to calculate the β coefficient under the null hypothesis of non-serially correlated errors. Run OLS of y_t on x_t and note the estimated coefficient $\hat{\beta}_0$.

- Evaluate ϵ_t and z_t at the restricted values, $\hat{\beta}_0$ and $\hat{\phi}_0 = 0$. Thus,

$$\hat{\epsilon}_t = y_t - x_t'\hat{\beta}_0.$$

$$z_t = \begin{bmatrix} y_{t-1} - x_{t-1}'\hat{\beta}_0 \\ x_t' \end{bmatrix}. \quad (4)$$

- Finally regress $\hat{\epsilon}_t$ on z_t or run the regression:

$$\hat{\epsilon}_t = \underbrace{y_{t-1} - x'_{t-1}\hat{\beta}_0 \text{ and } x'_t}_{z_t}.$$

- Take R^2 from this regression and test:

$$LM = TR^2 \sim^a \chi^2(1).$$

1.6 Test for Co-Integration

- $y_t \sim I(1)$ is cointegrated with $x_t \sim I(1)$ iff there exists a vector α, such that,

$$y_t - \alpha' x_t = u_t \sim I(0).$$

- To check, regress y_t on x_t and save the residuals, $\hat{u}_t$. $\hat{u}_t$ needs to be $I(0)$ if y_t and x_t are to be cointegrated.

- Regress:

$$\Delta\hat{u}_t = \phi_0 \hat{u}_t + \epsilon_t + \underbrace{\sum_j \phi_j \Delta\hat{u}_{t-j}}_{Other\ Lags}.$$

The other lags added on make this an "Augmented Dicky Fuller Test."

- Test:

$$H_0 : \phi_0 = 0 \quad \text{(NOT Cointegrated)}.$$
$$H_1 : \phi_0 < 0 \quad \text{(Cointegrated)}.$$

- Why? If $\phi_0 = 0$, then (ignoring the other lags),

$$\Delta\hat{u}_t = \epsilon_t \sim I(0).$$

Thus $\Delta\hat{u}_t \sim I(0)$ which means that $\hat{u}_t$ is not $I(0)$. We don't know for sure that it's $I(1)$, but it may be.

- So compute $\dfrac{\hat{\phi}_0}{SE(\hat{\phi}_0)}$ and reject H_0 if it is less than the critical value in the MacKinnon tables. (NOTE: we can't use the DF tables here).

- More on Tests for Co-Integration. If u_t is a stationary $AR(1)$, then x and y will be conintegrated. Consider the equation for u_t:

$$u_t = \phi u_{t-1} + \epsilon_t, \quad \epsilon_t \sim iid \quad |\phi| < 1.$$

Thus,

$$u_t - u_{t-1} = \phi u_{t-1} - u_{t-1} + \epsilon_t.$$

$$\Delta u_t = u_{t-1}(\phi - 1) + \epsilon_t.$$

$$\Delta u_t = u_{t-1}\gamma + \epsilon_t, \quad \gamma = \phi - 1.$$

Note that u_t is stationary if $|\phi| < 1$ and Δu_t is stationary if $|\gamma| < 1$. Thus test:

$H_0 : |\gamma| = 0 \Longrightarrow \Delta u_t \sim I(0) \Rightarrow \phi = \gamma + 1 = 1 \Rightarrow u_t \sim I(1) \Rightarrow (x, y)$ are NOT conintegrated.

$H_1 : |\gamma| < 0 \Longrightarrow \phi = \gamma + 1 < 1 \Rightarrow u_t \sim I(0) \Rightarrow (x, y)$ are conintegrated.

1.7 Properties of Standard Processes

1.7.1 AR(1)

- Model:

$$y_t = \phi y_{t-1} + \epsilon_t.$$

- Yields:

$$E[y_t] = 0.$$

$$Var(y_t)\Bigg|_{Stationary} = E[y_t^2] = \frac{\sigma^2}{1 - \phi^2}.$$

$$Cov(y_t, y_{t-1}) = \frac{\phi\sigma^2}{1 - \phi^2}.$$

$$Cov(y_t, y_{t-s}) = \frac{\phi^s\sigma^2}{1 - \phi^2}.$$

- Stationary if $|\phi| < 1$.
- In another form after backward substitution:

$$y_t = \phi^s y_{t-s} + \sum_{j=0}^{s-1} \phi^j \epsilon_{t-j}.$$

1.7.2 MA(1)

- Model:

$$y_t = \theta\epsilon_{t-1} + \epsilon_t.$$

- Yields:

$$E[y_t] = 0.$$

$$Var(y_t) = E[y_t^2] = \sigma^2(1 + \theta^2).$$

$$Cov(y_t, y_{t-1}) = \theta\sigma^2.$$

$$Cov(y_t, y_{t-s}) = 0 \;\forall\; s > 1.$$

1.7.3 Random Walk

- Model:

$$\Delta y_t = \epsilon_t \Longrightarrow y_t = y_{t-1} + \epsilon_t.$$

- Yields:

$$E[y_t] = 0.$$

$$Var(y_t) = t\sigma^2.$$

1.7.4 Random Walk with Drift

- Model:

$$\Delta y_t = \alpha + \epsilon_t \Longrightarrow y_t = \alpha + y_{t-1} + \epsilon_t.$$

- Yields:

$$E[y_t] = \alpha t.$$

$$Var(y_t) = t\sigma^2.$$

1.8 Asymptotic Distribution of $\hat{\beta}$

- We would first like to check that the $\hat{\beta}$ estimator is consistant.

$$\hat{\beta} = (X'X)^{-1}X'y.$$

$$\hat{\beta} = (X'X)^{-1}X'[X\beta + \epsilon] = \beta + (X'X)^{-1}X'\epsilon.$$

$$\hat{\beta} - \beta = (X'X)^{-1}X'\epsilon.$$

- Or rewriting:

$$\hat{\beta} - \beta = \left[\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right]^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} x_i \epsilon_i\right].$$

- Taking the probability limit:

$$plim(\hat{\beta} - \beta) = plim\left[\left[\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right]^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} x_i \epsilon_i\right]\right].$$

  By Slutsky's Theorem:

$$plim(\hat{\beta} - \beta) = \left[plim\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right]^{-1}\left[plim\frac{1}{n}\sum_{i=1}^{n} x_i \epsilon_i\right].$$

  Since $x_i x_i'$ is an *iid* sequence, and $E[x_i x_i'] = \Sigma_{xx}$, then by the Weak Law of Large Numbers (WLLN),

$$plim \frac{1}{n} \sum_{i=1}^{n} x_i x_i' = \Sigma_{xx}.$$

Also, $x_i \epsilon_i$ is an *iid* sequence with $E[x_i \epsilon_i] = 0$ because of independence. Thus, by the WLLN,

$$plim \frac{1}{n} \sum_{i=1}^{n} x_i \epsilon_i = 0.$$

- Thus, substituting these last two equations in,

$$plim(\hat{\beta} - \beta) = \Big[\Sigma_{xx}\Big]^{-1} \Big[0\Big] = 0.$$

- Thus, $\hat{\beta}$ is a consistant estimator. Note that the two substitutions could have also been done via the ergodic theorem which does not rely on *iid* but rather on stationarity and limited memory. (*iid* processes are always Ergodic).

1.9 Lagged Dependents and Serial Correlated Errors

- Consider the following model:

$$y_t = \gamma y_{t-1} + x_t' \beta + u_t, \quad u_t = \phi u_{t-1} + \epsilon_t.$$

- Compute $E[y_{t-1} u_t]$.

$$\begin{aligned}
E[y_{t-1} u_t] &= E[(\gamma y_{t-2} + x_{t-1}' \beta + u_{t-1})(\phi u_{t-1} + \epsilon_t)]. \\
&= \gamma \phi E[y_{t-2} u_{t-1}] + \beta \phi E[x_{t-1}' u_{t-1}] + \phi E[u_{t-1}^2] + \gamma E[y_{t-2} \epsilon_t] + \beta E[x_{t-1}' \epsilon_t] + E[u_{t-1} \epsilon_t]. \\
&= \gamma \phi E[y_{t-2} u_{t-1}] + \beta \phi E[x_{t-1}' u_{t-1}] + \phi E[u_{t-1}^2]. \\
&= \gamma \phi E[y_{t-1} u_t] + \beta \phi E[x_t' u_t] + \phi E[u_t^2]. \\
&= \frac{\beta \phi E[x_t' u_t] + \phi E[u_t^2]}{1 - \gamma \phi}. \\
&= \frac{\phi E[u_t^2]}{1 - \gamma \phi}. \\
&= \frac{\phi Var(u_t)}{1 - \gamma \phi}. \\
&= \frac{\phi \frac{\sigma^2}{1 - \phi^2}}{1 - \gamma \phi}. \\
&= \frac{\phi \sigma^2}{(1 - \phi^2)(1 - \gamma \phi)} \neq 0.
\end{aligned}$$

- Thus, in this situation, OLS is biased. If ϕ is unknown, use the C-O transformation to estimate the true parameter, β. If ϕ is unknown (more realistic), then rewrite model by lagging, multiplying by ϕ and subtracting which gives us iid errors. Then do MLE on resulting equation. Note that running OLS will work here but you won't be able to distinguish the parameter ϕ from the β parameter. MLE is equivalent (linearly) to OLS and allows us to discern to two.

1.10 Final Review before Exam

- Variances.
$$Var(\hat{\beta}) = E\Big[\hat{\beta} - \beta\Big]^2.$$
- Roots.
$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t.$$
$$y_t - \phi_1 L y_t - \phi_2 L^2 y_t = \epsilon_t.$$
Roots solve:
$$Z^2 - \phi_1 Z - \phi_2 = 0.$$
Thus,
$$\Delta y_t = \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \epsilon_t.$$
$$\Delta y_t - \phi_1 \Delta L y_t - \phi_2 \Delta L y_t = \epsilon_t.$$
$$(1 - \phi_1 L - \phi_2 L)\Delta y_t = \epsilon_t.$$
$$(1 - \phi_1 L - \phi_2 L)(y_t - y_{t-1}) = \epsilon_t.$$
$$(1 - \phi_1 L - \phi_2 L)(1 - L) y_t = \epsilon_t.$$
Roots solve:
$$(Z^2 - \phi_1 Z - \phi_2)(Z^2 - Z) = 0.$$
Note that $Z = 1$ satisfies this equation.
- $TSS = RSS + ESS$.
$$R^2 = \frac{ESS}{TSS} = \frac{\hat{y}'\hat{y}}{y'y} = \frac{TSS - RSS}{TSS} = \frac{y'X\hat{\beta}}{y'y}.$$
$$s^2 = \frac{RSS}{N-K} = \frac{y'y - y'X\hat{\beta}}{N-K}.$$
$$s^2_{ML} = \frac{RSS}{N}.$$
- Wald.
$$\frac{\Big(R\hat{\beta} - q\Big)' \Big(Rs^2(X'X)^{-1}R'\Big)^{-1} \Big(R\hat{\beta} - q\Big)}{r} \sim F(r, N-K).$$

- Likelihood:

$$\Pi_t^T\left[\frac{1}{\sqrt{2\pi\sigma^2}}exp\Big(-\frac{\epsilon_t}{2\sigma^2}\Big)\right].$$

- LM test.

$$I(\psi) = -E\Big[\frac{\partial^2 log\ L}{\partial\psi\partial\psi'}\Big].$$

$$Var(\hat{\psi}) = I(\hat{\psi})^{-1} = -E\Big[\frac{\partial^2 log\ L}{\partial\psi\partial\psi'}\Big]^{-1}.$$

Under $R(\psi) = 0$,

$$LM = \frac{\partial log\ L}{\partial\psi'} I(\hat{\psi})^{-1} \frac{\partial log\ L}{\partial\psi} \sim \chi^2(q).$$

Note that $I(\psi)$ is block diagonal with the block corresponding to β as:

$$\frac{1}{\hat{\sigma}_0^2}\sum z_t z_t'.$$

Thus we can also write the LM stat (seen from regression of ϵ_t on z_t and computing TR^2):

$$LM = \frac{\Big(\sum z_t\epsilon_t\Big)'\Big(\sum z_t z_t'\Big)^{-1}\Big(\sum z_t\epsilon_t\Big)}{\hat{\sigma}_0^2}.$$

Note:

$$\frac{\partial log\ L(\hat{\psi}_0)}{\partial\psi'} = \frac{1}{\hat{\sigma}_0^2}(\sum z_t\epsilon_t)'.$$