Macroeconomics I
Lent Term

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1 Week 1: 14 Jan - 18 Jan

1.1 Dynamic Economic Models
An Introduction to Linear Differential Equations

- Consider the first order ordinary linear differential equation,

$$\frac{dX}{dt} = \dot{X}(t) = \lambda X(t) + f(t).$$

- The $X(t)$ part of the equation is simply the level of X, while the $f(t)$ function is often referred to as a forcing function. $f(t)$ drives X. We will eventually model the money growth rate as $f(t)$ and show its influence on the growth rate of inflation, $X(t)$.

- Now rearrange our equation:

$$\dot{X}(t) - \lambda X(t) = f(t).$$

Note that now we have the endogenous variable, $X(t)$ on the left and the exogenous variable, $f(t)$ alone on the right. Now multiply everything through by $e^{-\lambda t}$,

$$e^{-\lambda t}\dot{X}(t) - e^{-\lambda t}\lambda X(t) = e^{-\lambda t} f(t).$$

Note that the left hand side can be rewritten as follows,

$$\frac{d}{dt}\left[X(t)e^{-\lambda t}\right] = e^{-\lambda t} f(t).$$

Now take the definite integral of both sides evaluated from t_1 to t_2,

$$\int_{t_1}^{t_2} \frac{d}{dt}\left[X(t)e^{-\lambda t}\right] dt = \int_{t_1}^{t_2} e^{-\lambda t} f(t) dt.$$

$$X(t_2)e^{-\lambda t_2} - X(t_1)e^{-\lambda t_1} = \int_{t_1}^{t_2} e^{-\lambda t} f(t) dt.$$

Solve for $X(t_2)$,

$$X(t_2)e^{-\lambda t_2} = X(t_1)e^{-\lambda t_1} + \int_{t_1}^{t_2} e^{-\lambda t} f(t) dt.$$

$$X(t_2) = \frac{X(t_1)e^{-\lambda t_1} + \int_{t_1}^{t_2} e^{-\lambda t} f(t) dt}{e^{-\lambda t_2}}.$$

$$X(t_2) = X(t_1)e^{-\lambda t_1 + \lambda t_2} + \int_{t_1}^{t_2} e^{-\lambda t + \lambda t_2} f(t) dt.$$

$$X(t_2) = X(t_1)e^{\lambda(t_2 - t_1)} + \int_{t_1}^{t_2} e^{-\lambda(t - t_2)} f(t) dt.$$

This equation for $X(t_2)$ is simply the way that $X(t)$ behaves in the future given what has happened in t_1. We can also solve for $X(t_1)$ which is a little harder to justify. It is basically the level of X today given that people's perceptions of what is going to happen in the future are what that matters in determining the levels of X today. Solving using a similar method,

$$X(t_1) = X(t_2)e^{-\lambda(t_2-t_1)} - \int_{t_1}^{t_2} e^{-\lambda(t-t_1)} f(t)dt.$$

- Note that if $\lambda < 0$,

$$X(t_2) \to \int_{-\infty}^{t_2} f(t)e^{\lambda(t_2-t)}dt \text{ as } t_1 \to -\infty.$$

- Note that if $\lambda > 0$,

$$X(t_1) \to -\int_{t_1}^{\infty} f(t)e^{-\lambda(t-t_1)}dt \text{ as } t_2 \to \infty.$$

1.2 A Model of Inflation

- Let $\dot{\pi}(t)$ be the growth rate of inflation. Thus,

$$\dot{\pi}(t) = \frac{\pi(t) - \mu(t)}{\alpha},$$

with $\alpha > 0$. $\pi(t)$ is clearly the level of inflation and $\mu(t)$ is the growth of the money supply and will act as our forcing equation in this differential equation.

- Skipping the intermediate steps for the time being, this yields,

$$\pi(t) = \alpha^{-1} \int_0^{\infty} \underbrace{\mu(t+s)}_{\textit{Stream of Money Growth}} \overbrace{e^{-s\alpha^{-1}}}^{\textit{Discount Rate}} ds.$$

1.3 A Model of Asset Prices

- Let $\dot{Q}(t)$ be the rate of change of an asset price. Thus,

$$R = \frac{\pi}{Q} + \frac{\dot{Q}}{Q},$$

where $\dfrac{\dot{Q}}{Q}$ is the proportional rate of change or the rate of capital accumulation. π in this setting is the level of dividends paid out, and Q is the level of asset price. R can be thought of as the "safe" rate of return (Government Bond rate). Rewriting this expression,

$$\dot{Q}(t) = RQ - \pi.$$

And again, this will be developed later, so skipping the steps, the solution turns out to be,

$$Q(t) = \int_0^\infty \pi(t+s)e^{-sR}ds \text{ if } R \text{ is constant.}$$

$$Q(t) = \int_0^\infty \pi(t+s)e^{-\int_0^s R(t+u)du}ds \text{ otherwise.}$$

Note that the first expression is identical to the inflation model solution.

1.4 Extended Model of Inflation: Cagan's Adaptive Expectations Model

- Consider the following model of the demand for money balances,

$$\frac{M_t}{P_t} = y_t e^{-\alpha \hat{P}_t^e}.$$

So we have the demand for real money balances on the left hand side and income and expected inflation on the right hand side.

- A few notes on notation:

$$ln(P_t) = p_t.$$

$$\frac{d}{dt}[ln(P_t)] = \frac{d}{dt}[p_t] = \dot{p}_t.$$

OR!!

$$\frac{d}{dt}[ln(P_t)] = \frac{dP_t/dt}{P_t} = \frac{\dot{P}_t}{P_t}.$$

Thus,

$$\dot{p}_t = \frac{\dot{P}}{P}.$$

Finally,

$$\dot{p}_t^e = \frac{\dot{P}_t^e}{P_t^e} = \hat{P}_t^e \equiv \text{The proportional rate of change.}$$

- So to simplify the analysis, set $y_t = \bar{y} = 1$. Let $m_t = ln(M_t)$ and $p_t = ln(P_t)$. Thus,

$$\frac{M_t}{P_t} = y_t e^{-\alpha \hat{P}_t^e}.$$

$$\frac{M_t}{P_t} = e^{-\alpha \hat{P}_t^e}.$$

$$\frac{ln(M_t)}{ln(P_t)} = ln(e^{-\alpha \hat{P}_t^e}).$$

$$m_t - p_t = -\alpha \hat{P}_t^e = -\alpha \dot{p}_t^e = -\alpha \pi_t^e.$$

Differentiating with respect to t,

$$\dot{m}_t - \dot{p}_t = -\alpha\dot{\pi}_t^e.$$

Let $\mu_t = \dot{m}_t$ and let $\pi_t = \dot{p}_t$, so:

$$\mu_t - \pi_t = -\alpha\dot{\pi}_t^e.$$

$$\mu_t = \pi_t - \alpha\dot{\pi}_t^e.$$

Or,

$$\pi_t = \mu_t + \alpha\dot{\pi}_t^e.$$

- Now consider a seperate statement called the "Adaptive Expectations Equation":

$$\dot{\pi}_t^e = \theta(\pi_t - \pi_t^e).$$

- Substituting in,

$$\dot{\pi}_t^e = \theta(\mu_t + \alpha\dot{\pi}_t^e - \pi_t^e).$$

- More on this next lecture.

1.5 More on the Inflation under Adaptive Expectations Model

- If $\lambda < 0$, this implies that the equation is "stable." Thus, as time evolves, as $t_2 \to \infty$, we have exponential convergence to 0.
- If $\lambda > 0$, this implies that the equation is "unstable." Thus, as $t \to \infty$, the exponential explodes to infinity.
- Consider again the equation which relates the growth of the money supply to the inflation rate and the expected growth rate of inflation:

$$\mu_t = \pi_t - \alpha\dot{\pi}_t^e.$$

But how is the last term determined, that is, how is the expected growth rate of inflation, $\dot{\pi}_t^e$, determined? Hence the Adaptive Expectations Equation:

$$\dot{\pi}_t^e = \theta(\pi_t - \pi_t^e).$$

Substituting in,

$$\dot{\pi}_t^e = \theta(\mu_t + \alpha\dot{\pi}_t^e - \pi_t^e).$$

And rearranging,

$$\dot{\pi}_t^e - \theta\alpha\dot{\pi}_t^e = \theta(\mu_t - \pi_t^e).$$

$$\dot{\pi}_t^e(1-\theta\alpha) = \theta(\mu_t - \pi_t^e).$$

$$\dot{\pi}_t^e = \frac{\theta(\mu_t - \pi_t^e)}{(1-\theta\alpha)}.$$

- So if θ is high, this means the people are quick to react to what is going on around them so they quickly adjust their inflationary expectations.
- If θ is low, this means that people adjust their inflationary expections in line with the way in which they adjust for errors they have made previously so therefore, the adjustment process is slow.
- The important part of this is that if θ is high, the equation becomes explosive (unstable). To get stability, a desirably property, θ must be low which implies that people have to be sort of ignorant about changes around them.
- So does this second implication make the whole theory of adaptive expections less desirable? Well, consider an alternative: Rational expectations.

1.6 Inflation under Rational Expectations

- Again we have the same equation as before,

$$\mu_t = \pi_t - \alpha\dot{\pi}_t^e.$$

- But this time, we introduce an equation of a different form than the adaptive expectations equation. The rational expectations equation is as follows:

$$\pi_t^e = \pi_t.$$

  Which implies,

$$\dot{\pi}_t^e = \dot{\pi}_t \ \forall \ t \neq t_0.$$

  Notice that in this equation, there is no mysterious θ term. People form expectations from the model. They form expectations as if they already know the underlying model of inflation.

- Substituting in again,

$$\mu_t = \pi_t - \alpha\dot{\pi}_t.$$

$$\pi_t - \mu_t = \alpha\dot{\pi}_t.$$

$$\dot{\pi}_t = \frac{\pi_t - \mu_t}{\alpha}.$$

- The solution to this problem (which we will approach later),

$$\pi_t = \alpha^{-1} \int_0^\infty \underbrace{\mu_{t+s}}_{\textit{Money Growth Stream}} \overbrace{e^{-\alpha^{-1}s}}^{\textit{Discount Rate}} ds.$$

- Note that this implies that inflation rates are forward looking (via rational expectations) instead of backward looking (via adapative expectations). Thus, π_t is the present discounted value of the monetary expectations for the future.

- Graphically, [**G-1.1**] it is nice to look at the germanic countries in the interwar period as they experienced a period of hyperinflation. A pure monetarist would expect that the inflation rate would move exactly with the growth rate of money as they believe that monetary growth causes inflationary fluctuations. In reality we don't see that as the inflation rate falls quickly at one point below the growth rate of the money supply. This must be because people are revising their estimates of monetary growth (via rational expectations: they know the underlying model) and inflation rate falls ahead of monetary growth cuts.

1.7 The Budget Deficit and Inflation Sargent and Wallace

- Fiscalists like Sargent and Wallace argue that the real driving force behind inflation is not necessarily the growth of the money supply, but rather the level of government debt. The fiscal side is the fundamental root of inflationary fluctuations. If you want to stabilize inflation, balance the budget. The model proceeds as follows:

- Consider the following money demand function:

$$\frac{M}{P} = ky.$$

At this equation alone, noting that money demand depends only on income, the monetarists argument is strong. Since y and k are both out of the government's control, if M is increasing at some rate, in order to maintain equality, P must also be "zooming" up with it. But there's more ...

- Assume a constant rate of income growth such that:

$$\hat{y} = \frac{\dot{y}}{y} = n.$$

- Now consider the following interest rate equation:

$$r = (1 - \tau_1)R - \hat{P} = \rho.$$

Where r is the real interest rate and is equal to the after tax nominal interest rate, R, less inflation. We call this quantity, ρ.

- Finally, we have a fiscal equation:

$$\frac{\dot{M} + \dot{B}}{P} = d + (1 - \tau_1)Rb.$$

So on the right hand side, first of all, we have government expenditure. d is defined as the primary government deficit or,

$$d = g - \tau_0 - \tau_1 y.$$

The other part of expenditure is the after tax value of the payment to the general public. (ie, b is the level of real government debt and the government must pay the interest, R, on that debt back to the public.)

On the left hand side, we have government finances. The government can generate finances in two ways: borrowing money or printing money. If they borrow money, $B =$ Bonds, they increase the real government debt, b. If they print money, $M = M$oney, they expand the money supply which based on the money demand equation, will drive up prices. Thus,

$$\frac{\dot{B}}{P} \equiv \text{ The rate of increase in real government debt.}$$

$$\frac{\dot{M}}{P} \equiv \text{ The rate of increase in real money supply.}$$

- So fiscalists look at this last equation and say that it is the debt that really matters for stabilizing the money supply. Putting on the monetary breaks now will only cause the current debt to cost more in the future. Cutting the money supply, ie, raising the interest rate, drives up Rb.

2 Week 2: 21 Jan - 25 Jan

2.1 Multivariate Systems of Differential Equations

- Consider the following differential equation as we had from previous analysis,

$$\dot{X}(t) = AX(t) + f(t).$$

With A as a nXn matrix. If A was diagonal: the reduces to the previous case where the system can be completely decoupled. If A is not diagonal, then the different entries in X co-move. We call this "Extensive Feedback." The system cannot be decoupled as before.

- However, even if A is not diagonal, it might be, and usually is, diagonalizable. If it is, then,

$$A = V\Lambda V^{-1},$$

where,

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

V is therefore orthogonal.

- Substituting in,

$$\dot{X}(t) = AX(t) + f(t).$$
$$\dot{X}(t) = V\Lambda V^{-1}X(t) + f(t).$$

And premultiplying by V^{-1},

$$V^{-1}\dot{X}(t) = \Lambda V^{-1}X(t) + V^{-1}f(t).$$

Now let $X^* = V^{-1}X(t)$ and let $f^*(t) = V^{-1}f(t)$. Thus,

$$\dot{X}^*(t) = \Lambda X^*(t) + f^*(t).$$

This system is decoupled: we have a collection of equations, each with the same properties as before. At the end of the analysis of the system, however, it will be necessary to multiply back through by V to regain the economic significance:

$$X(t) = VX^*(t).$$

- We will need to look at the eigenvalues of Λ to determine the stability of the system. For this course, we will only consider real eigenvalues so the sign of those values will be sufficient for determining stability.

2.2 A Multivariate Economic Model: Blanchard's Model: Stock Market and Real Activity

- There are 4 things that we would like to get from the following analysis:
  - 1) We will determine the interaction between the commodity and financial markets.
  - 2) This model will show that asset markets react quickly to economic shocks.
  - 3) The dynamics and expectations in this model will help us understand common every day relationships.
  - 4) We'll find some variables are "Jumpers" and some are "Crawlers." This just means that some of the variables will be continuous and others, not.

- Lets begin with the demand for goods and service, y^d:

$$y^d = \delta Q + g.$$

  Here we have aggregate demand, maybe GNP or something, depending positively on government spending, or the "fiscal surplus", and since $\delta > 0$, it depends positively on the index of the wealth in the stock market, Q.

- Now consider the supply side, or the productive capacity of the economy, y. We model it such that the productive capacity reacts to the surplus or excess of demand in the economy:

$$\dot{y} = \alpha(y^d - y).$$

  So, if we assume $\alpha > 0$, then if aggregate demand, y^d, is greater than the supply in the economy, y, then $\dot{y} > 0$ and the productive capacity grows in response. Will it grow all the way to reduce the differential to zero, it doesn't matter. It's just the relationship that matters. Think of $\dot{y}$ as the change in the level of GDP.

- Now we turn to the money market:

$$M = \gamma_0 + \gamma_1 y - \gamma_2 R.$$

  Where the demand for money balances, M, depends positively on GDP, y, and negatively on the interest rate, R. (Assuming γ_0, γ_1, $\gamma_2 > 0$.) Note also that we write money demand and the interest rate in nominal terms. This is because we are assuming that prices are fixed so it wouldn't make any difference if we used real terms.

- Now consider the return on capital,

$$R_k = \underbrace{\frac{\pi}{Q}}_{Dividend\,Yield\,Ratio} + \underbrace{\frac{\dot{Q}}{Q}}_{Rate\,of\,Stock\,Market\,Appreciation}.$$

Here, π is the flow of dividends. Hence, we can think of the right side of the equation as the two forms of payoff from investment, dividends and capital gains. Rearranging this equation, we get,

$$\dot{Q} = QR_k - \pi.$$

- Due to this last equation, it is clear that the rate of return on capital must equal the rate of return in the economy (otherwise there would be an arbitrage opportunity): Assuming risk neutrality,

$$R = R_k^e.$$

Where R is the rate of return on government bonds, say.

- Now we have a dividend equation,

$$\pi = \beta_0 + \beta_1 y.$$

So dividends depend positively on the aggregate output, y, or GDP. We can think of this as a behaviorial equation from the point of view of the managers. As their firms make profits when production is high, they pay them back out in the form of dividend payments to stock holders.

- Now, assuming myopic perfect foresight,

$$\dot{Q}^e = Q.$$

The expected rate of change of the wealth in the stock market is equal to the wealth in the stock market currently. (??)

- Consider this equation for $\dot{Q}$ above, rewritten here:

$$\dot{Q} = QR_k - \pi,$$

we have shown that if R_k is constant, then solutions to the differential equation will be of the form:

$$Q(t) = \int_0^\infty \pi(t+s)e^{-R_k s}ds,$$

Or that the level of wealth in the stock market is equal to the present discounted value of the flow of dividends from now out to time infinity. More generally, this can be written, (without the constraint on R_k),

$$Q(t) = \int_0^{\infty} \pi(t+s) e^{-\int_0^s R_k(t+u)du} ds,$$

So from this we can see that $Q = f(\pi)$ and from before we know that $\pi = f(y)$, thus, $Q = f(y)$.

- Now consider the following equation which we get by substituting in the expression for R_k and π into:

$$\dot{Q} = QR_k - \pi.$$

$$\dot{Q} = Q \underbrace{\left[\frac{\gamma_0 + \gamma_1 y - M}{\gamma_2} \right]}_{R_k} - \underbrace{\left[\beta_0 + \beta_1 y \right]}_{\pi}.$$

- Note that the "State" of a dynamical system is the minimum set of variables that define all parameters in equilibrium.

- Taking the differential of the last expession,

$$d(\dot{Q}) = R^* dQ + Q^* \frac{\gamma_1}{\gamma_2} dy - \beta_1 dy.$$

As we have let R^* and Q^* represent the equilibrium values of each of these quantities.

- Thus, since $\dot{Q}^* = 0$ by definition, then $d(\dot{Q}^*) = 0$ as well. Thus,

$$\dot{Q} = R^*(Q - Q^*) + (Q^* \frac{\gamma_1}{\gamma_2} - \beta_1)(y - y^*).$$

- Substituting the demand for goods, $y^d = \delta Q + g$ into the supply equation, $\dot{y} = \alpha(y^d - y)$, yields:

$$\dot{y} = \alpha(\delta Q + g - y).$$

Taking the differential:

$$d(\dot{y}) = \alpha\delta dQ + \alpha dg - \alpha dy.$$

Noting that $dg = 0$,

$$d(\dot{y}) = \alpha\delta dQ - \alpha dy.$$

And performing a similar linearization operation as before:

$$\dot{y} = \alpha\delta(Q - Q^*) - \alpha(y - y^*).$$

- Thus the two equations, $\dot{Q}$ and $\dot{y}$ describe the "state" of the dynamical system around the region of Q^* and y^*. In matrix form:

$$\begin{bmatrix} \dot{Q} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} R^* & Q^*\frac{\gamma_1}{\gamma_2} - \beta_1 \\ \alpha\delta & -\alpha \end{bmatrix} \begin{bmatrix} Q - Q^* \\ y - y^* \end{bmatrix}. \tag{1}$$

2.3 More on the Stock Market / Real Activity Model

- Consider again the minimum state vector:

$$\begin{bmatrix} \dot{Q} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} R^* & Q^*\dfrac{\gamma_1}{\gamma_2} - \beta_1 \\ \alpha\delta & -\alpha \end{bmatrix}}_{A} \begin{bmatrix} Q - Q^* \\ y - y^* \end{bmatrix}. \tag{2}$$

- The A matrix, will help us determine the stability of the system.
- All the systems that we will deal with will be 2x2 systems so we will only have to look at the sign of two eigenvectors to determine stability.
- In this example, note the determinant of A:

$$|A| = -\alpha R^* - \alpha\delta[Q^*\frac{\gamma_1}{\gamma_2} - \beta_1].$$

Since R^* and Q^* are both greater than zero, and all the greek coefficients are assumed to be positive, then

$$|A| < 0 \text{ if } \beta_1 \text{ is not too large.}$$

- Note that β_1 is the coefficient on output in the dividend equation. If β_1 is large, this means that a slight fluctuation in the level of output causes large fluctuations in dividend yields. This makes the stock market increase causing AD to increase. Then this mechanism feeds on itself through a sort of feedback system so everything begins to grow exponentially. So to get stability, we must have β_1 not too large. This means that the system is stable if $|A| < 0$.
- Considering again the state vector involving the A matrix, the state vector itself is just defined as (Q, y). The values of these two variables determine the entire system. Note that Q, the wealth of the stock market, is a "jumper" variable. Asset prices respond quickly and discontinuously to external changes. The other state variable, y, GDP, is a "crawler." The time path of y is continuous. Noting the equation for $\dot{y}$, AD could jump discontinuously, but y only changes gradually and cannot make sudden movements.

- Determining the eigenvalues of A, we find that one is positive and one is negative. Therefore we have a situation where the dynamical system of equations will begin to look like a saddle-path.

2.4 Overview of Comparative Dynamics

- Step 0.) Get to $\dot{Z} = \psi(Z - Z^*) + \xi$.
- Step 1.) Establish configuration element by element for $\dot{Z} = 0$, ie, the long run steady state.
- Step 2.) Clarify jumpers and crawlers.
- Step 3.) Find saddle-point saddlepath and eyeball local dynamics.
- Step 4.) Move 'em on out. In other words, use your model to do policy experiments and introduce shocks to see how the model behaves.

2.5 Back to Model Analysis

- Note that Crawler type variables will be backward looking and they will line up with the stable, negative eigenvalue.
- Jumper variables will usually be more forward looking so they are unstable and line up with the positive eigenvalue.
- If $\beta_1 < Q^* \frac{\gamma_1}{\gamma_2}$, ie, it's not too large, we can draw the situation in a graph of y, Q space. See notes. [**G-2.1**]
- The only point that both variables are stable is at the point labeled E in the graphs where both $\dot{y}$ and $\dot{Q}$ equal 0.
- [**G-2.2**] To determine the slopes of these lines consider the state equation involving $\dot{y}$ and $\dot{Q}$. If we know that these variables are not changing, the left hand side of these equations must be zero. Looking on the right, the only way to get those expressions to sum to zero is if Q and y are inversely related when $\dot{Q} = 0$ and positively related when $\dot{y} = 0$.
- [**G-2.3**] Note that we can also determine the stability of these lines by looking at a small perturbation away from one of the equilibrium lines, without changing the other, and seeing what happens. It turns out that Q is unstable, in that movements away from $\dot{Q} = 0$ are followed by further movements away. y is stable. So the stock market is unstable ... reference irrational exuberance.
- We can then plot the eigenvectors of the system to determine boundries for the system. See notes again for this. The unstable eigenvalue is associated with the "U" eigenvector and if we are ever on that line, the system will move us away from equilibrium. The opposite is true about the stable, "S" eigenvector.

- **[G-2.4]** Finally, we can move to the last part of the analysis and shock the system to see what happens. Suppose there is an unexpected policy shock and the money supply increases. This will shift up the $\dot{Q} = 0$ line. In order to get us back to equilibrium, we would have to be exactly on the stable eigenvector path at the original level of y. Otherwise, the system behaves unstably.

3 Week 3: 28 Jan - 1 Feb

3.1 Output and Stock Market Model

- [**G-3.1**] We again consider an unanticipated increase in the money supply. Recall the state space that we found previously with the $\dot{y} = 0$ sloping upwards in (y, Q) space and the $\dot{Q} = 0$ line sloping downwards. See graph in notes.

- The eventual effect, as can be seen by the algebra, is that $\dot{Q} = 0$ will shift out and we will reach a higher level of both y and Q in equilibrium.

- At the instant that the money supply increases, the value of the stock market jumps up to point c in the graph. y is a "crawler" variable and cannot "jump." Thus y must not move discontinuously but we know that we eventually have to reach R in the long run. Thus, the jump in Q must be exactly to point c, the eigenvector associated with the stable y variable.

- Once at c, we slowly move down the eigenvector towards R, raising y, but decreasing Q a little from its peak at c. Thus, in effect, the jump in stock valuations overshot the equilibrium level and thus there is some falling back that occurs eventually.

- The graph in the notes [**G-3.2**] also shows the movement of the key variables. Note that M, the money supply, shifts up at time t_0 and then remains constant. At the same time, as we said, Q jumps up and then falls back eventually to some equilibrium level above its initial level. y slowly rises after t_0 but is continuous. The last variable we plotted was the nominal interest rate. As soon as the money supply is increased, in order for the money market to stay in equilibrium, since money demand is initially constant, R must fall. So we see an immediate jump downwards. Once y starts to rise, money demand shifts to the right. This raises the interest rate gradually (again, via the money market) as is shown in the graph. [**G-3.3**] Where R ends up, (above or below its initial level) all depends on the elasticity of money demand with respect to income, m_y.

- This movement of R back up, after its initial jump down, also reflects the movement in Q back down to its equilibrium level. As R rises, dividend gains start to slow down (due to a high discount rate). Thus, the movement of Q from E to c to R is really not the stock market overshooting or overreacting, but really is just a standard logical economic movement.

- There are two other examples in the lecture notes on page 8.4.

3.2 Uncovered Interest Parity - UIP

- Consider equation (3), UIP:

$$\underbrace{R_t - R_t^*}_{Interest\ Rate\ Differential} = \underbrace{\frac{(e_{t+1}^e - e_t)}{e_t}}_{Expected\ Rate\ of\ Depreciation}.$$

Where e_t is the price of foreign currency in domestic units. ie, it costs 80 pence to "buy" a dollar. Note this is the opposite of how they are quoted in the US. If $e_{t+1} - e_t > 0$ or the price of foreign currency is expected to be higher in the future, this clearly means that the domestic currency is expected to depreciate. If $R_t^* > R_t$, then the rate of return on a foreign asset is higher than the domestic rate of return. Thus the left hand side of equation 3 is negative. Thus for UIP to hold, then the right side must also be negative which implies $e_{t+1} - e_t < 0$. Thus the exchange rate is expected to be lower in the future. Thus, the price of foreign currency will go down in the future, ie, the domestic currency is expected to appreciate. For instance, this implies that if an investor buys a foreign asset at time t, and earns the interest on that asset denominated in foreign currency, when he cashes out of the investment at time $t+1$, even though he has earned a higher return abroad, when he transfers that money back into domestic currency, he loses out because his domestic currency has appreciated. Thus, if the differential in the interest rate is exactly canceled out by the expected change in the exchange rate, which UIP implies, then investors are indifferent from investing at home or abroad.

- How did we arrive at equation (3)? Well, the value of a 1 pound investment in a domestic asset is equal to $1 + R_t$. The value of a 1 pound investment in a foreign asset is $\frac{(1+R_t^*)e_{t+1}^e}{e_t}$. Setting these equal, we find:

$$1 + R_t = (1 + R_t^*)\frac{e_{t+1}^e}{e_t}.$$

  And because $\frac{1+R_t}{1+R_t^*} \simeq 1 + R_t - R_t^*$, substituting in, we get the UIP.

- For future models, we will use the UIP and we will write it in the following form:

$$\dot{e} = R - R^*.$$

  Or the rate of change in the exchange rate is equal to the interest rate differential between domestic and foreign assets.

3.3 Setup of the Dornbusch Model

- The model takes the standard Mundell-Fleming apparatus of Section 2 and introduces uncovered interest parity coupled with sluggish price adjustment. All variables except R and R^* are in logs.

- First the money market:

$$m - p = \alpha y - \beta R.$$

With m as the log of money demand, we have the demand for real money balances on the left hand side. On the right, we have positive dependence on income, y and negative dependence on the interest rate. As the interest rate rises, people would rather hold non-money assets. (Assuming $\alpha, \beta > 0$).

- Aggregate Demand:

$$y^d = -\gamma(R - \dot{p}) + \delta(e - p).$$

So, we have AD depending negatively on the real interest rate. (IS curve slopes downwards). And AD depends positively on the real exchange rate. I suppose that as e rises, or the price of foreign currency rises, or the domestic currency depreciates, it is cheaper for foreigners to buy our goods. Thus since net exports are a determinant of AD, a depreciation of the currency leads to higher demand.

- Phillips curve relationship:

$$\dot{p} = \phi(y - \bar{y}).$$

Or the inflation rate is related to how far current actual income is from permanent income.

- And finally, UIP:

$$\dot{e} = R - R^*.$$

- More next time...

3.4 More on the Dornbusch Model

- Refer to last lectures notes for setup of this model. The one wrinkle in the analysis is the phillips curve relation which must hold at equilibrium.

- Consider the 4 equations again :

$$m - p = \alpha y - \beta R.$$
$$y = -\gamma(R - \dot{p}) + \delta(e - p).$$
$$\dot{p} = \phi(y - \bar{y}).$$
$$\dot{e} = R - R^*.$$

Substituting the phillips curve relationship into the AD relationship,

$$y = -\gamma(R - \phi(y - \bar{y})) + \delta(e - p).$$
$$y = -\gamma(R - \phi y + \phi\bar{y}) + \delta(e - p).$$
$$y = -\gamma R + \gamma\phi y - \gamma\phi\bar{y} + \delta e - \delta p.$$
$$y - \gamma\phi y + \gamma R = -\gamma\phi\bar{y} + \delta e - \delta p.$$
$$y(1 - \gamma\phi) + \gamma R = -\gamma\phi\bar{y} + \delta e - \delta p.$$

Taking this last equation and the equation for money demand (rearranged),

$$\alpha y - \beta R = m - p.$$
$$y(1 - \gamma\phi) + \gamma R = -\gamma\phi\bar{y} + \delta e - \delta p.$$

These two equations are what we are looking for because we have the endogenous variables, y and R on the left and the exogenous variabes: m,p,$\bar{y}$, and e on the right. Thus, we can write this system in matrix form:

$$\begin{bmatrix} \alpha & -\beta \\ 1-\gamma\phi & \gamma \end{bmatrix} \begin{bmatrix} y \\ R \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -\delta & \delta & 0 & -\gamma\phi \end{bmatrix} \begin{bmatrix} p \\ e \\ m \\ \bar{y} \end{bmatrix}. \tag{3}$$

- Multiplying the right side matrices together,

$$\begin{bmatrix} \alpha & -\beta \\ 1-\gamma\phi & \gamma \end{bmatrix} \begin{bmatrix} y \\ R \end{bmatrix} = \begin{bmatrix} -p+m \\ -\delta p + \delta e - \gamma\phi\bar{y} \end{bmatrix}. \tag{4}$$

- Now we can solve for y and R using Cramer's Rule: We now have a system in the form $AX = B$. So to find y, take the determinant of A with the corresponding column of A replaced by B and divide through by the determinant of A. Thus,

$$y = \frac{\begin{vmatrix} -p+m & -\beta \\ -\delta p + \delta e - \gamma\phi\bar{y} & \gamma \end{vmatrix}}{\begin{vmatrix} \alpha & -\beta \\ 1-\gamma\phi & \gamma \end{vmatrix}}. \tag{5}$$

$$y = \frac{(-p+m)\gamma - \beta(\delta p - \delta e + \gamma\phi\bar{y})}{\alpha\gamma + \beta(1-\gamma\phi)}.$$

Let $\psi = \dfrac{1}{\alpha\gamma + \beta(1-\gamma\phi)}$. Thus,

$$y = \psi\Big[(-p+m)\gamma - \beta(\delta p - \delta e + \gamma\phi\bar{y})\Big].$$

$$y = \psi\Big[\gamma(m-p) + \beta\delta(e-p) - \beta\gamma\phi\bar{y}\Big].$$

- Now let's do the same for R:

$$R = \frac{\begin{vmatrix} \alpha & -p+m \\ 1-\gamma\phi & -\delta p + \delta e - \gamma\phi\bar{y} \end{vmatrix}}{\begin{vmatrix} \alpha & -\beta \\ 1-\gamma\phi & \gamma \end{vmatrix}}. \tag{6}$$

$$R = \frac{\alpha(-\delta p + \delta e - \gamma\phi\bar{y} - (-p+m)(1-\gamma\phi)}{\alpha\gamma + \beta(1-\gamma\phi)}.$$

Let $\psi = \dfrac{1}{\alpha\gamma + \beta(1-\gamma\phi)}$. Thus,

$$R = \psi\Big[(\gamma\phi - 1)(m-p) - \alpha\delta p + \alpha\delta e - \alpha\gamma\phi\bar{y}\Big].$$
$$R = \psi\Big[(\gamma\phi - 1)(m-p) + \alpha\delta(e-p) - \alpha\gamma\phi\bar{y}\Big].$$

- Now we look at steady state properties, $\dot{e} = \dot{p} = 0$. From the phillips curve relation and the UIP equation, this implies that $R = R^*$ and $y = \bar{y}$. Thus the money demand equation and the AD equation become, (solving for p and $e - p$),

$$p = m - \alpha\bar{y} + \beta R^*.$$
$$e - p = \frac{(\bar{y} + \gamma R^*)}{\delta}.$$

Hence, the model exhibits classical neutrality in the long run (??). This last item does not appear to be necessary for the comparative dynamics.

- Now we substitute our two derived equation for R and y (the two equations that solved the system), into the phillips curve and UIP relationships. First Phillips:

$$\dot{p} = \phi(y - \bar{y}).$$

$$\dot{p} = \phi\Bigg[\psi\Big[\gamma(m-p) + \beta\delta(e-p) - \beta\gamma\phi\bar{y}\Big] - \bar{y}\Bigg].$$

$$\dot{p} = \phi\Bigg[\psi\gamma(m-p) + \psi\beta\delta(e-p) - \psi\beta\gamma\phi\bar{y} - \bar{y}\Bigg].$$

$$\dot{p} = \phi\psi\gamma(m-p) + \phi\psi\beta\delta(e-p) - \phi\psi\beta\gamma\phi\bar{y} - \phi\bar{y}.$$

$$\dot{p} = \phi\psi\gamma m - \phi\psi\gamma p + \phi\psi\beta\delta e - \phi\psi\beta\delta p - \bar{y}(\phi\psi\beta\gamma\phi + \phi).$$

$$\dot{p} = \phi\psi\gamma m + p(-\phi\psi\gamma - \phi\psi\beta\delta) + \phi\psi\beta\delta e - \bar{y}(\phi\psi\beta\gamma\phi + \phi).$$

$$\dot{p} = \phi\psi\gamma m + p(-\phi\psi)(\gamma + \beta\delta) + \phi\psi\beta\delta e + \bar{y}(-\phi)(\psi\beta\gamma\phi + 1).$$

- And then the UIP:

$$\dot{e} = \psi\Big[(\gamma\phi - 1)(m - p) + \alpha\delta(e - p) - \alpha\gamma\phi\bar{y}\Big] - R^*.$$

$$\dot{e} = \psi(\gamma\phi - 1)(m - p) + \psi\alpha\delta(e - p) - \psi\alpha\gamma\phi\bar{y} - R^*.$$

$$\dot{e} = \psi(\gamma\phi - 1)m - \psi(\gamma - \phi)p + \psi\alpha\delta e - \psi\alpha\delta p - \psi\alpha\gamma\phi\bar{y} - R^*.$$

$$\dot{e} = \psi(\gamma\phi - 1)m + p(-\psi(\gamma - \phi) - \psi\alpha\delta) + \psi\alpha\delta e - \psi\alpha\gamma\phi\bar{y} - R^*.$$

$$\dot{e} = \psi(\gamma\phi - 1)m + p\psi(1 - \gamma\phi - \alpha\delta) + \psi\alpha\delta e - \psi\alpha\gamma\phi\bar{y} - R^*.$$

- And we can write this system in matrix form as follows:

$$\begin{bmatrix} \dot{p} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} -\phi\psi(\gamma + \beta\delta) & \phi\psi\beta\delta \\ \psi(1 - \gamma\phi - \alpha\delta) & \psi\alpha\delta \end{bmatrix} \begin{bmatrix} p \\ e \end{bmatrix} + \begin{bmatrix} \phi\psi\gamma & 0 & \phi\psi(\beta + \alpha\gamma) \\ \psi(\gamma\phi - 1) & -1 & \alpha\gamma\psi\phi \end{bmatrix} \begin{bmatrix} m \\ R^* \\ \bar{y} \end{bmatrix}. \tag{7}$$

**Note the inconsistancy in the $\bar{y}$ term of the $\dot{p}$ equation. I cannot seem to find where I went wrong.

- The system is saddle-path stable iff the determinant of the first matrix is negative. Call this determinant, Δ. Thus,

$$\Delta = -\phi\psi^2(\alpha\delta)(\gamma + \beta\delta) - \psi\phi\beta\delta\phi(1 - \gamma\phi - \alpha\delta).$$

$$\Delta = -\phi\psi\delta.$$

- Δ is negative as long as $\psi > 0$ because both ϕ and δ are assumed to be positive coefficients. Recall that ψ is the determinant of the last huge matrix we calculated:

$$\psi = \frac{1}{\alpha\gamma + \beta(1 - \gamma\phi)}.$$

  So $\psi > 0$ iff $\gamma\phi < 1$. γ is the coefficient on the real interest rate in the AD equation. ϕ is the coefficient that measures inflation's sensitivity to differences in supply and demand (via the Phillips curve).

- Now lets determine the slope of the stationaries. Consider the $\dot{p}$ equation and let $\dot{p} = 0$. Now consider only the p and e terms since the rest will drop out when we take the derivative.

$$0 = p(-\phi\psi)(\gamma + \beta\delta) + \phi\psi\beta\delta e.$$

  Thus,

$$p(\phi\psi)(\gamma + \beta\delta) = \phi\psi\beta\delta e.$$
$$e = \frac{p(\phi\psi)(\gamma + \beta\delta)}{\phi\psi\beta\delta}.$$

$$e = \frac{p(\gamma + \beta\delta)}{\beta\delta}.$$

$$\frac{de}{dp} = \frac{(\gamma + \beta\delta)}{\beta\delta} > 1 > 0.$$

  And the slope of the $\dot{e} = 0$ line is found the same way:

$$0 = p\psi(1 - \gamma\phi - \alpha\delta) + \psi\alpha\delta e.$$
$$-p\psi(1 - \gamma\phi - \alpha\delta) = \psi\alpha\delta e.$$
$$e = \frac{-p\psi(1 - \gamma\phi - \alpha\delta)}{\psi\alpha\delta}.$$
$$\frac{de}{dp} = \frac{-(1 - \gamma\phi - \alpha\delta)}{\alpha\delta} = \frac{\gamma\phi + \alpha\delta + 1}{\alpha\delta} <> 0.$$

- Now, we can just assume $\alpha\delta + \gamma\phi < 1$, so the $\dot{e} = 0$ line slopes downwards.

- From here, we have to determine two more things ... 1) Determine if each of the two variables stable or unstable. 2) Draw in the eigenvectors of each of them to determine the saddle path areas. The stable eigenvector will form our saddle path stable line. We can then do dynamic experiments and try to figure out what happens when there is an unanticipated permanent increase or decrease in the money supply. [**G-3.4**]

- **[G-3.5]**, **[G-3.6]** In the case of a permanent and unexpected monetary cut, we get the exchange rate overshooting (because of the slope of the saddle path stable line. Why? A cut in the money stock requires a long run appreciation, but domestic prices are sluggish so real balances fall. Hence domestic interest rates rise above foreign rates, producing a positive interest differential which must be offset by an expected depreciation.

4 Week 4: 4 Feb - 8 Feb

4.1 Time Consistency Kyland & Prescot

- "The problem of time inconsistency arises when an economic agent, say the government, has an incentive to promise to take some action ex-ante, but to take some other action ex-post, even though there may be no extraneous shocks or new information that might otherwise lead it to alter its plans."
- "However, other agents will anticipate that the agent will renege on its promises in this fashion and will alter their behaviour accordingly, leading to an inferior outcome."
- Evidence suggests at least in the UK, that when ever an incumbent party won an election, it came at the same time that the central bank had been cutting interest rates. The opposite is also seen where incumbents will lose elections when the central bank happens to be raising interest rates. Suspicious.
- We also saw in the data that central bank independence has basically nothing to do with long term economic growth, but it has a lot to do with the growth rate of inflation. Particularly, the more independent the central bank is, the less inflation growth that is seen.

4.1.1 The Positive Theory of Inflation

- Consider the following model. First an expectations augmented aggregate supply curve:

$$y_t = \alpha(p - E_{t-1}[p_t]) = \alpha(\pi_t - E_{t-1}[\pi_t]).$$

  Where $\pi_t = p_t - p_{t-1}$.

- And consider a loss function for the economy as follows:

$$L_t = \pi_t^2 + \lambda(y_t - y^*)^2.$$

  Where λ is the relative weight put on deviations from the long run equilibrium level (growth rate) of output. Since societies dislike instability, deviations, positive or negative, are viewed as bad for the economy.

- So there are two ways to model the ways in which policy makers make decisions: Commitment or Discretion.
- Commitment:
  - Set π_t before $E_{t-1}[\pi_t]$ is developed. Assume the central bank has perfect control over the rate of inflation.

- Thus, clearly, to minimize the loss function, L, the central bank should make sure $\pi_t^c = 0$, or the growth rate of inflation is zero. By doing this however, they have committed themselves to this strategy so people's expections, being rational, will expect just this, so $E_{t-1}[\pi_t] = 0$. Thus the aggregate supply function becomes:

$$y_t = \alpha(\pi_t - E_{t-1}[\pi_t]) = \alpha(0 - 0) = 0.$$

  Noting that this is a growth rate of output so we are basically just saying that output will not grow under these circumstances. Thus our loss function:

$$L_t = \pi_t^2 + \lambda(y_t - y^*)^2 = 0 + \lambda(0 - y^*)^2 = \lambda y^{*2}.$$

$$L_t^c = \lambda y^{*2}.$$

- Discretion:
  - Here, we set π_t optimally after $E_{t-1}[\pi_t]$ is formed. So we wait to figure out what people are going to expect inflation to be, and then we set policy instruments to determine the actual level of inflation such that it minimizes L. Note how the commitment strategy is not a Nash equilibrium because the central bank will have the incentive to renege on its strategy. Here we seek to minimize the loss function:

$$min_{\pi_t} L_t = \pi_t^2 + \lambda\left[\alpha(\pi_t - E_{t-1}[\pi_t]) - y^*\right]^2.$$

  - FOC:

$$2\pi_t + 2\lambda\alpha\left[\alpha(\pi_t - E_{t-1}[\pi_t]) - y^*\right] = 0.$$

$$\pi_t + \lambda\alpha\left[\alpha(\pi_t - E_{t-1}[\pi_t]) - y^*\right] = 0.$$

$$\pi_t + \lambda\alpha^2\pi_t - \lambda\alpha^2 E_{t-1}[\pi_t] - \alpha\lambda y^* = 0.$$

$$(1 + \lambda\alpha^2)\pi_t = \lambda\alpha^2 E_{t-1}[\pi_t] + \alpha\lambda y^*.$$

  - In this setting, we have been taking $E_{t-1}[\pi_t]$ as GIVEN. But now look at the optimisation. Suppose we were following the commitment regime and therefore, $E_{t-1}[\pi_t] = 0$. Then $\pi_t = \dfrac{\alpha\lambda y^*}{1 + \lambda\alpha^2} \neq E_{t-1}[\pi_t]$. Thus the mere announcement of intentions to have zero inflation is NOT credible. So the only credible annoucement is one in which $E_{t-1}[\pi_t] = \pi_t$. Thus our FOC becomes:

$$(1 + \lambda\alpha^2)\pi_t = \lambda\alpha^2\pi_t + \alpha\lambda y^*.$$
$$\pi_t^d = \alpha\lambda y^*.$$

And the loss function becomes:

$$L_t = \pi_t^2 + \lambda\left[\alpha(\pi_t - E_{t-1}[\pi_t]) - y^*\right]^2 = (\alpha\lambda y^*)^2 + \lambda y^{*2}.$$

$$L_t^d = \lambda y^{*2}(\alpha^2\lambda + 1).$$

- Now comparing the loss functions under commitment and discretion:

$$L_t^d - L_t^c = \lambda y^{*2}(\alpha^2\lambda + 1) - \lambda y^{*2} = \lambda y^{*2} + \lambda^2\alpha^2 y^{*2} - \lambda y^{*2} = \lambda^2\alpha^2 y^{*2} = (\lambda\alpha y^*)^2.$$

- This extra term is the deadweight loss associated with the government's inability to commit to zero inflation ex-ante.
- So what all this says is that when we impose a restriction on the model, such as the commitment approach, the result is actually less efficient which is counter-intuitive. The reasoning behind this involves the time inconsistancy as explained earlier.
- Ways to get around time inconsistancies include things like reputation effects that come into play say when governments are in power for a longer period of time and care about their reputations. It is also possible to impose a constitutional rule forbidding reneging at a later date. This however, puts restraints on policy stabilization when shocks hit the economy.
- Two other examples of time inconsistancy. First, the online MP3 music market. The Napsters of the world allow societies to drive the marginal cost of production down to zero which is socially efficient. However, ex-ante, we need to create a positive incentive for musicians to make their music and profit from it. Clearly, having artists stop making music would be a less optimal solution than actually paying for CDs at a record store. The internet and its free flowing information has caused P=MC=0.
- Another example is taxation. Governments can finance deficits with taxes on fixed factors of production. Factors like labor might be persuaded to easily substitute leisure for work if an income tax was imposed, but they could impose the tax on fixed (immovable) inputs like capital. However, ex-ante, people will also expect this and invest less in capital which is also not a first best solution. This creates a dead weight loss for society.

4.2 More on Time Inconsistancy

- Consider the problem of pharmacuticals. The supply of drugs once they have been developed costs only a few cents per pill. However, the cost involved in developing the drugs are huge. Thus, ex-post, 3rd world countries and others complain that if the marginal cost of supplying a pill is only a few cents, why not charge these countries that really need them this price. Ex-ante however, the drug makers have incurred huge fixed costs that need to be recovered.

4.2.1 Extension to Positive Inflation Theory

- Consider the following extension where we now allow exogenous shocks to enter the supply equation.

$$y_t = \alpha(\pi_t - E_{t-1}[\pi_t]) + \epsilon_t, \quad \epsilon_t \; iid \; (0, \sigma^2).$$

- And consider the standard loss function as before:

$$L_t = \pi_t^2 + \lambda(y_t - y^*)^2.$$

- Under the discretion case, central bankers minimize the loss function taking expected inflation rates as given. Thus, minimize:

$$\pi_t^2 + \lambda(\alpha(\pi_t - \underbrace{E_{t-1}[\pi_t]}_{Given}) + \epsilon_t - y^*)^2.$$

- The first order condition with respect to π_t,

$$2\pi_t + 2\lambda\alpha(\alpha(\pi_t - E_{t-1}[\pi_t]) + \epsilon_t - y^*) = 0.$$
$$\pi_t + \lambda\alpha(\alpha(\pi_t - E_{t-1}[\pi_t]) + \epsilon_t - y^*) = 0.$$
$$\pi_t + \lambda\alpha^2\pi_t - \lambda\alpha^2 E_{t-1}[\pi_t] + \lambda\alpha\epsilon_t - \lambda\alpha y^* = 0.$$
$$\pi_t(1 + \lambda\alpha^2) = \lambda\alpha^2 E_{t-1}[\pi_t] - \lambda\alpha\epsilon_t + \lambda\alpha y^*.$$
$$\pi_t(1 + \lambda\alpha^2) = \alpha^2\lambda E_{t-1}[\pi_t] + \alpha\lambda(y^* - \epsilon_t).$$

- Taking expectations:

$$E_{t-1}[\pi_t](1 + \lambda\alpha^2) = \alpha^2\lambda E_{t-1}[\pi_t] + \alpha\lambda(y^* - E_{t-1}[\epsilon_t]).$$
$$E_{t-1}[\pi_t] = \alpha\lambda(y^* - E_{t-1}[\epsilon_t]).$$
$$E_{t-1}[\pi_t] = \alpha\lambda y^*.$$

  Note that ϵ_t is a shock that is known to the central bankers at the time they are making their policy decision.

This last expression is the same as the expected inflation under the discretionary case with NO shocks. Now substitute this expression back into the FOC, to solve for the optimal level of inflation:

$$\pi_t(1+\lambda\alpha^2) = \alpha^2\lambda E_{t-1}[\pi_t] + \alpha\lambda(y^* - \epsilon_t).$$
$$\pi_t(1+\lambda\alpha^2) = \alpha^2\lambda\alpha\lambda y^* + \alpha\lambda(y^* - \epsilon_t).$$
$$\pi_t(1+\lambda\alpha^2) = \alpha^3\lambda^2 y^* + \alpha\lambda y^* - \alpha\lambda\epsilon_t.$$
$$\pi_t(1+\lambda\alpha^2) = y^*(\alpha^3\lambda^2 + \alpha\lambda) - \alpha\lambda\epsilon_t.$$
$$\pi_t(1+\lambda\alpha^2) = y^*\alpha\lambda(\alpha^2\lambda + 1) - \alpha\lambda\epsilon_t.$$
$$\pi_t = \alpha\lambda y^* - \underbrace{\frac{\alpha\lambda\epsilon_t}{1+\alpha^2\lambda}}_{Shock\ Term}.$$

- Note that on average (or taking expectations), $\pi_t^e = \alpha\lambda y^*$. So when there is a positive shock to the economy, $\epsilon_t > 0$, then the central bank sets the inflation rate to be lower so as to "lean against the wind," and stabilize the movements of key economic variables.

- Now consider the loss function under discretion:

$$L_t = \pi_t^2 + \lambda(y_t - y^*)^2.$$

Substituting,

$$L_t = (\alpha\lambda y^* - \frac{\alpha\lambda\epsilon_t}{1+\alpha^2\lambda})^2 + \lambda(y_t - y^*)^2.$$

Taking expectations:

$$E[L_t] = \alpha^2\lambda^2 y^{*2} + \frac{\alpha^2\lambda^2 E[\epsilon_t^2]}{(1+\alpha^2\lambda)^2} + \lambda(E[y_t^2] - 2E[y_t y^*] + y^{*2}).$$

And somehow ...

$$E[L_t^D] = \lambda\left[(1+\alpha^2\lambda)y^{*2} + \sigma^2/(1+\alpha^2\lambda)\right].$$

- If the government could commit to follow a rule, it would choose to follow:

$$\pi_t = -\frac{\alpha\lambda\epsilon_t}{1+\alpha^2\lambda}.$$

However, if the private sector does not observe ϵ_t, it is difficult to see how private agents can monitor whether the authorities are sticking to this rule.

- ALSO, the government could really be hawks on inflation and set $\pi_t = 0$ no matter what. Then the loss function collapses to:

$$L_t = \lambda(\epsilon_t - y^*)^2.$$

Taking expectations,

$$E[L_t^R] = \lambda(\sigma^2 + y^{*2}).$$

- A goverment would prefer descretion to a rule on inflation iff:

$$E[L_t^D] < E[L_t^R].$$

Or,

$$y^{*2} < \underbrace{\sigma^2/(1+\alpha^2\lambda)}_{Volatility\ of\ Shocks}.$$

Note this equation makes perfect intuitive sense. If the economy if very volatile, then most likely this inequality will be satisfied and the government would prefer to have discretion in making policy decisions to react when shocks hit. If their hands were tied, then there wouldn't be much they could do to soften the landing after an economic shock. If the economy is more stable, $\sigma^2 \approx 0$, then the government would prefer a more hawkish central bank keeping inflation very low.

- OR, there is one final possibility that actually results in the first best outcome. DELEGATION. The government could install a central banker that has different preferences than the rest of the economy, say with loss funcion:

$$L_t = \pi_t^2 + \mu(y_t - y^*)^2$$

with $\mu < \lambda$. Then this conservative central banker places more weight on the inflation rate staying in check then on output fluctuations. In the end we find that a weight μ^* should be optimally somewhere between zero and λ to minimize the loss function for the economy. See graph in notes. **[G-4.1]**

- Finally, installing contracts for central bankers is also a way to make sure that the optimal weights are put on inflation and output fluctuations in the loss function. Consider the following loss function:

$$L_t = \pi_t^2 + \lambda(y_t - y^*)^2 - (\phi + \psi\pi_t).$$

Here we have a side payment being made to the central banker that is a function of the economy wide inflation rate. When we solve the model for the optimal rate of inflation we get:

$$\pi_t = \frac{-\alpha\lambda\epsilon_t}{1+\alpha^2\lambda}.$$

Which is precisely the same as the commitment strategy so this is the first best solution.

5 Week 5: 11 Feb - 15 Feb

5.1 The Life Cycle Model

- Consider a two period world where a consumer receives initial endowment wealth, A, as well as labor income Y_1 in period 1 and Y_2 in period 2. The consumer can borrow or lend at rate r, and thus has savings S. Consumers consume C_1 in period 1 and C_2 in period 2.

- Consumers seek to maximize with the choice of C_1 and C_2:

$$U(C_1, C_2).$$

Subject to a constraint:

$$\underbrace{A + Y_1 + \frac{Y_2}{1+r}}_{PDV\ of\ Stream\ of\ Income} \geq \underbrace{C_1 + \frac{C_2}{1+r}}_{PDV\ of\ Stream\ of\ Consumption}.$$

- Thus on the left we have the present discounted value of total income and on the right, the present discounted value of consumption over the two periods. $\frac{1}{1+r}$ is the relative price of consumption in period 2 where the numeraire is the price of consumption in period 1.

- This interpretation of the budget constraint is called the "Integrated Budget Constraint."

- Another interpretation. Consider the consumer who maximizes:

$$U(C_1, C_2),$$

subject to:

$$\underbrace{A + Y_1}_{Income\ Period\ 1} \geq C_1 + S,$$

and:

$$\underbrace{(1+r)S + Y_2}_{Income\ Period\ 2} \geq C_2.$$

Here we have just rewritten the budget constraint above using two periods instead of one. We have also included the savings term, S. Note that if you multiply the first constraint by $(1+r)$:

$$(1+r)A + (1+r)Y_1 = (1+r)C_1 + (1+r)S.$$

$$(1+r)A + (1+r)Y_1 - (1+r)C_1 = (1+r)S.$$

And substitute it into the second constraint:

$$(1+r)S + Y_2 \geq C_2.$$
$$(1+r)A + (1+r)Y_1 - (1+r)C_1 + Y_2 \geq C_2.$$
$$A + Y_1 - C_1 + \frac{Y_2}{1+r} \geq \frac{C_2}{1+r}.$$
$$A + Y_1 + \frac{Y_2}{1+r} \geq C_1 + \frac{C_2}{1+r}.$$

Which is exactly the Integrated Budget constraint.

- This can be referred to as a "Sequence of Optimizations."
- Thus, the integrated budget constraint assumes there are "No Credit Restrictions" and thus the S is substituted out. Thus we can have both positive and negative savings (borrowing). Depending on the utility function, it might be better to borrow now and consume, than to wait and consume tomorrow.
- Setting up the Lagrangian:

$$\mathbb{L} = U(C_1, C_2) - \lambda\Big[C_1 + \frac{C_2}{1+r} - A - Y_1 - \frac{Y_2}{1+r}\Big].$$

- FOCs:

$$\frac{\partial \mathbb{L}}{\partial C_1} = U_1(C_1, C_2) - \lambda = 0.$$
$$\frac{\partial \mathbb{L}}{\partial C_1} = U_2(C_1, C_2) - \frac{\lambda}{1+r} = 0.$$

- Thus,

$$\frac{U_1(C_1, C_2)}{U_2(C_1, C_2)} = 1 + r.$$

So the ratio of marginal utilities in each period, or the intertemporal marginal rate of substitution, is equal to the gross interest rate.

- This last equation is about dynamics. The slope of the Marginal Utilities must equal the gross interest rate at equilibrium configurations. A higher r makes it more worthwhile for a consumer to refrain from consumption today. (ie they could save their income from period 1 and have even more in period 2). However, that is just the substitution effect. There is also an income effect: A higher r reduces $\frac{Y_2}{1+r}$, or the PDV of future income. So overall, usually the change in consumption pattern is ambiguous from a change in the interest rate. Thus,

$$C_r <> 0.$$

- However, we do know something else:

$$C_1 = C(r, W),$$

where $W = A + Y_1 + \frac{Y_2}{1+r}$. If C_1 is normal, then $C_W > 0$. (ie an increase in total wealth will increase consumption.)

- So 3 key points to take away from all of this:
  - 1) Changes in r affect the dynamic allocation of consumption and savings.
  - 2) Using one budget constraint assumes no credit restrictions. (When studying imperfect credit markets, we'll have to use the alternative interpretion.)
  - 3) C_1 is dependent on r and W, but C_1 does not depend on the time path of labor income. If Y_1 is much less than Y_2, it doesn't matter as only the PDV of total wealth will matter for decisions regarding consumption.

5.1.1 Life Cycle Model: Multiple Periods

- The consumer has a known death date, T, and faces a known income sequence $\{Y_s\}$, $s = t \ldots T$. The consumer's problem is now written as follows. Maximize with respect to C_s and A_s:

$$\underbrace{\sum_{s=t}^{T} \frac{U(C_s)}{(1+\delta)^{s-t}}}_{PDV\ of\ Utility\ Flow}.$$

So now we are maximizing the discounted flow of utility by choosing optimal consumption in each period, s, and also optimal asset holdings.

- $\frac{1}{1+\delta}$ is the subjective discount factor. Otherwise consumer's rate of dissatisfaction from utility in the future.

- The budget constraints are written as follows:

$$\underbrace{(1+r)A_{s-1}}_{Non-labor\ Wealth\ with\ interest} + Y_s - C_s \geq A_s \ \ \forall\ s.$$

And,

$$A_T \geq 0.$$

Thus, in the first constraint(s), we have income less expenditure on the left hand side. Income is composed of current labor income, Y_s, and the interest augmented assets

from last period. We subtract from income, consumption in period s, C_s. All this has to be greater than A_s, or the total asset holdings in period s. NOTE: this constraint is defined recursively. It is NOT just one constraint but rather $T - t$ constraints, one for each period.
The second constraint is called the "Transversality Constraint." It states that assets at the time of death but be non-negative. Meaning you can't pile on the debt during your life and just die without paying any of it back.

- Thus, the Lagrangian:

$$\mathbb{L} = \sum_{s=t}^{T} \frac{U(C_s)}{(1+\delta)^{s-t}} - \sum_{s=t}^{T} \lambda_s \Big[A_s - (1+r)A_{s-1} - Y_s + C_s \Big] + \mu A_T.$$

- Notes that the first multiplier, λ_s, is actually a sequence of multipliers. FOCs:

$$\frac{\partial \mathbb{L}}{\partial C_s} \Rightarrow \frac{U'(C_s)}{(1+\delta)^{s-t}} - \lambda_s = 0 \text{ for } s = t \dots T.$$

$$\frac{\partial \mathbb{L}}{\partial A_s} \Rightarrow -\lambda_s + (1+r)\lambda_{s+1} = 0 \text{ for } s = t \dots T-1.$$

$$\frac{\partial \mathbb{L}}{\partial A_T} \Rightarrow \mu - \lambda_T = 0.$$

KEY POINT: When taking that partial with respect to A_s, the $-\lambda_s$ is clear, but the we also must take the partial with respect to A_{s-1} to get $(1+r)\lambda_{s+1}$ because of the recursive nature of the first constraint. Also this partial only involves times up to $T-1$ because the T^{th} time period is taken care of by the third FOC. Here the μ is clear, but the $-\lambda_T$ comes again from the recursive nature of the first constraint.

5.2 More on the Life Cycle Model in Multiple Periods

- Recall the first order conditions:

$$\frac{U'(C_s)}{(1+\delta)^{s-t}} - \lambda_s = 0 \text{ for } s = t \dots T.$$

$$-\lambda_s + (1+r)\lambda_{s+1} = 0 \text{ for } s = t \dots T-1.$$

$$\mu - \lambda_T = 0.$$

- Evaluate the first FOC at period t:

$$\frac{U'(C_t)}{(1+\delta)^{t-t}} = U'(C_t) = \lambda_t.$$

Thus, the first FOC evaluated at t divided by the first FOC evaluated at s:

$$\frac{U'(C_t)}{U'(C_s)/(1+\delta)^{s-t}} = \frac{\lambda_t}{\lambda_s}.$$

- Now, consider the right hand side of this last equation, we can use the second FOC to simplify it:

$$\frac{\lambda_t}{\lambda_s} = \frac{(1+r)\lambda_{t+1}}{(1+r)\lambda_{s+1}}.$$

$$= \frac{(1+r)^2\lambda_{t+2}}{(1+r)^2\lambda_{s+2}}.$$

$$= \frac{(1+r)^3\lambda_{t+3}}{(1+r)^3\lambda_{s+3}}.$$

$$\vdots$$

$$= \frac{(1+r)^{T-t}\lambda_T}{(1+r)^{T-s}\lambda_T} = (1+r)^{T-t-T+s} = (1+r)^{s-t}.$$

Thus,

$$\frac{U'(C_t)}{U'(C_s)/(1+\delta)^{s-t}} = (1+r)^{s-t}.$$

- Thus, this last equation is the "Intertemporal MRS" and is analogous to the 2 period case. It is basically the slope of the consumption profile in time.
- Next consider the lifetime budget constraint as follows:

$$W_t = \underbrace{(1+r)A_{t-1}}_{Non-human\ Wealth} + \underbrace{\sum_{s=t}^{T}\left[\frac{Y_s}{(1+r)^{s-t}}\right]}_{PDV\ labor\ income} = \underbrace{\sum_{s=t}^{T}\left[\frac{C_s}{(1+r)^{s-t}}\right]}_{PDV\ total\ consumption}.$$

Thus total wealth is the sum of non-human wealth and the present discounted value of all present and future labor income.

- Thus $C_t = C(r, W_t; T-t)$.

5.2.1 CRRA Case

- Consider the case where the utility function of the consumer has the special property of "Constant Relative Risk Aversion" or CRRA.

- Consider total wealth as the discounted flow of lifetime utility from consumption with $\beta \in (0, 1)$:

$$W = \sum_{t=0}^{\infty} \beta^t U(C_t).$$

- Define a utility function as follows: For all $\gamma > 0$,

$$U(C) = \frac{C^{1-\gamma} - 1}{1 - \gamma}.$$

  Thus,

$$U'(C) = C^{-\gamma}.$$

  And,

$$U''(C) = -\gamma C^{-\gamma-1}.$$

- Proposition 1: As $\gamma \longrightarrow 1$, $U(C) \longrightarrow ln(C)$.
  Pf. Note that when $\gamma = 1$, $U(C) = \frac{0}{0}$, but otherwise, for $\gamma \neq 1$, the function is well-defined. Consider some special formula involving the limit:

$$\lim_{\gamma \to 1} U(C) = \frac{\lim_{\gamma \to 1} \frac{d}{d\gamma}(C^{1-\gamma})}{\lim_{\gamma \to 1} \frac{d}{d\gamma}(1-\gamma)} = \frac{\lim_{\gamma \to 1} \frac{d}{d\gamma}(C^{1-\gamma})}{-1}.$$

  For the top limit, consider: $C^{1-\gamma} = e^{ln(C^{1-\gamma})} = e^{(1-\gamma)ln(C)}$. Thus,

$$\frac{d}{d\gamma} e^{(1-\gamma)ln(C)} = e^{(1-\gamma)ln(C)} \cdot (-ln(C)) = e^{ln(C^{1-\gamma})} \cdot (-ln(C)) = C^{1-\gamma} \cdot (-ln(C)).$$

  Substituting this into the formula above,

$$\lim_{\gamma \to 1} U(C) = \lim_{\gamma \to 1} \frac{C^{1-\gamma} \cdot (-ln(C))}{-1} = \lim_{\gamma \to 1} ln(C)C^{1-\gamma} = ln(C).$$

  QED.

- Alternatively, the coefficient of relative risk aversion, $R(C)$ is defined as:

$$R(C) = -\frac{CU''(C)}{U'(C)} = -\frac{C(-\gamma C^{-\gamma-1})}{C^{-\gamma}} = \frac{\gamma C^{-\gamma}}{C^{-\gamma}} = \gamma.$$

  Hence CRRA.

- We also define the Intertemporal Elasticity of Substitution (IES) as γ^{-1}.

- Thus the consumer's problem becomes:

$$Max_{C_t, A_t} \sum_{t=0}^{\infty} \beta^t U(C_t).$$

  Subject to:

$$Y_t - C_t + (1+r)A_{t-1} \geq A_t.$$

  Therefore we get the following intertemporal MRS (following the same form as the previous two derivations):

$$\frac{\beta U'(C_{t+1})}{U'(C_t)} = (1+r)^{-1}.$$

  Noting that $U'(C) = C^{-\gamma}$, substituting in:

$$\frac{\beta C_{t+1}^{-\gamma}}{C_t^{-\gamma}} = (1+r)^{-1}.$$

$$\beta^{-1}(\frac{C_{t+1}}{C_t})^{\gamma} = (1+r).$$

  Taking logs,

$$\gamma ln(\frac{C_{t+1}}{C_t}) = ln(\frac{1+r}{\beta}).$$

$$\gamma lnC_{t+1} - \gamma lnC_t = ln(1+r) + ln(\beta).$$

$$\gamma lnC_{t+1} = \gamma lnC_t + ln(1+r) + ln(\beta).$$

$$lnC_{t+1} = lnC_t + \gamma^{-1}ln(1+r) + \gamma^{-1}ln(\beta).$$

- I'm not exactly sure where this takes us. See graph in lecture notes which shows how the consumption function is a smooth combination of intial borrowing early on, then savings in midlife, and finally dissavings after retirement until time T **when you get run over by a tractor**. The main idea is that consumption is a smoothed out function of life time wealth.

6 Week 6: 18 Feb - 22 Feb

6.1 Dynamic Programming

- Let Y' be a known function of Y such that:

$$Y' = \phi(Y).$$

In this context, the notation, Y', signifies, Y_{t+1} while Y signifies Y_t. The idea of defining some functional relationship between consequative occurances of a variable is called a "First Order Markov Assumption." We can do this WLOG.

- So consider the consumer's problem as we wrote it before:

$$Max_{C_t,A_t} \sum_{t=0}^{\infty} \beta^t U(C_t).$$

Subject to:

$$\underbrace{A_t + C_t}_{Allocation\ of\ Resources} \leq \underbrace{(1+r)A_{t-1} + Y_t}_{Total\ Resources}.$$

Note that we write the budget constraint in sequential form here which assumes there are no credit restrictions.

- We can also write this as a Dynamic Programme as follows:

$$V(A,Y) = Max_{\{C:A'+C\leq(1+r)A+Y\}} \ [U(C) + \beta V(A', \phi(Y))].$$

To explain: $V(A,Y)$ is called the value function and takes on arguments: A and Y which are called the "State" of the value function. $V(\cdot)$ is related to the utility function, though we really don't know its exact form. We maximize the utility gained from consumption, C, added to the discounted flow of utility from now until forever. Notice A', and $\phi(Y) = Y'$ are the values of A, non-human wealth, and Y, labor income, in a future period. Hence the recursive nature of the value function which gives the "program" its "dynamic" nature. Finally, we maximize over the budget constraint and in this form, we choose C. We could rewrite the program as follows:

$$V(A,Y) = Max_{\{A'\in[0,(1+r)A+Y]\}} \ [U([1+r]A + Y - A') + \beta V(A', \phi(Y))].$$

So here we substituted in for C in the objective function and now we maximize over A', such that A' is in the interval from 0, up to the total value of the resources available (in the earlier time period).

- Some notes on V. V is an unknown function that lives in some space of functions. Once we known V, we only have one optimization to do. $V(A,Y)$ is sometimes referred to as "Bellman's Equation" after Sir Henry Bellman the VII. In more general terms, V is defined on the state, x. Here $x = (A,Y)$.

- Finding V. There are really two methods for finding the functional form of V. 1) Guessing. There are two basic forms that V usually takes: Linear - Quadratic and log-linear/Cobb-Douglas. Otherwise, guessing will fail. 2) Iteration. This works by iterating to find a fixed point in the space of functions. Via the Contraction Mapping Theorem, these iterations will converge to the true V.

- Usage. Consider again the program derived above:

$$V(A,Y) = Max_{\{A'\}} \ [U([1+r]A + Y - A') + \beta V(A', \phi(Y))].$$

  Since we are maximizing over A', we can make use of the envelope theorem because $\dfrac{dA'}{dA} = 0$. Thus, we write:

$$V_1 = \frac{\partial V(A,Y)}{\partial A} = (1+r)U'(C).$$

  Notice that we need to take the partial of A' inside the U function and the A' inside the V function, but both derivatives will be zero by the envelope theorem.

- To complete the maximization, we will need the FOC of $V(A,Y)$ with respect to A'. Thus,

$$\frac{\partial V}{\partial A'} \Rightarrow -U'(C) + \beta V_1(A', \phi(Y)) = 0.$$

  Thus,

$$U'(C) = \beta V_1(A', \phi(Y)).$$

  Substituting in V_1 from above evaluated now at the next time period, C':

$$U'(C) = \beta(1+r)U'(C').$$

  Thus,

$$\beta \frac{U'(C')}{U'(C)} = (1+r)^{-1}.$$

  Or the intertemporal MRS discounted appropriately is equal to the inverse of $(1+r)$. A familiar result from our multiple period consumption optimisation problem last lecture.

6.1.1 Example: Dividend Paying Assets

- Consider the following maximization problem:

$$Max_{\{C_t, s_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(C_t).$$

  Subject to:

$$\underbrace{p_t s_{t+1} + C_t}_{Resource\ allocation} \leq \underbrace{(p_t + d_t)s_t}_{Resource\ sources}.$$

  So here we maximizing over consumption, C_t and stock holdings in the next time period. The objective function is the usual discounted flow of lifetime utility. The constraint is now written with the current values of stock holdings on the right: That is the current holdings, s_t, multiplied by the price of those holdings, p_t, added to the per unit dividend yield, d_t. On the left is what we can do with our resources. Either consume, C_t, or buy more stock, s_{t+1} at price p_t.

- Define an exogenous state: $x_t = \phi(x_{t-1})$. x_t can embody things like income, savings, weather, etc. Basically anything that can be considered as part of the state of the world. Again, this is a general Markov Process. Thus our two exogenous parameters, p_t and d_t can be defined as:

$$p_t = p(x_t).$$
$$d_t = d(x_t).$$

- We can then rewrite this problem in the dynamic programming form as follows:

$$V(s, x) = Max_{\{s' \leq (p+d)p^{-1}s\}} \ [U((p + d)s - ps') + \beta V(s', \phi(x))].$$

  Note here we have substituted out the C in the objective function and we are maximizing over the next period stock holdings being less than the expression shown ... I'm not sure why we have dropped the C from this equation but possibly because a solution to this maximization will also satisfy the maximization with C included.

- Via the envelope theorem,

$$V_1 = (p + d)U'(C).$$

- Via the FOC with respect to s',

$$-pU'(C) + \beta V_1(s', \phi(x)) = 0.$$

- Combining,

$$pU'(C) = \beta V_1(s', \phi(x)).$$
$$pU'(C) = \beta(p' + d')U'(C').$$

[Note that p and d are also dynamic so we have to update them to the current time period as well.]

$$\frac{p}{p' + d'}U'(C) = \beta U'(C').$$
$$\beta\frac{U'(C')}{U'(C)} = \frac{p}{p' + d'}.$$

This completes the example, though in Quah's notes there is an additional derivation which doesn't make very much sense at this point. This model will be referred to later so he might go through it then.

6.1.2 Uncertainty in the Dividend Paying Asset Example

- Consider the model as before:

$$Max_{\{C_t, s_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(C_t).$$

Subject to:

$$\underbrace{p_t s_{t+1} + C_t}_{Resource\ allocation} \leq \underbrace{(p_t + d_t)s_t}_{Resource\ sources}.$$

- But now the state is defined as $x_t = \phi(x_{t-1}, u_t)$ where $u_t \sim iid(0, \sigma^2)$. So we've added uncertainty to the state of the world. Not a whole lot changes. Consider the new value function:

$$V(s, x) = Max_{\{s' \leq (p+d)p^{-1}s\}} \Big[U((p + d)s - ps') + \beta E[V(s', \phi(x, u'))|x]\Big].$$

- Via the envelope theorem,

$$E[V_1|x] = E[(p + d)U'(C)|x].$$

- Via the FOC with respect to s',

$$-pU'(C) + \beta E[V_1(s', \phi(x, u'))|x] = 0.$$

- Combining,

$$pU'(C) = \beta E[V_1(s', \phi(x))|x].$$
$$pU'(C) = \beta E[(p' + d')U'(C')|x].$$

Again, here he substitutes back in the time indices and solves using limits though I still don't understand the limits. See his notes for more information.

6.2 Rational Expectations and Consumption

- See pages 10.6 thru 10.9 for this section.

- So far we have taken the sequence Y_s as known. Suppose we relax this and assume that both Y_s and r_s are uncertain. The consumer's optimization problem must now be written:

$$Max_{\{C_s, A_s\}} \ E_t\left[\sum_{s=t}^{T} \frac{U(C_s)}{(1+\delta)^{s-t}}\right].$$

Subject to:

$$\begin{cases} (2a) & A_s \leq (1+r)A_{s-1} + Y_s - C_s \qquad s = t, \ldots, T \\ (2b) & A_T \geq 0 \end{cases} \tag{8}$$

Substituting in the budget constraint, we can rewrite the optimisation as:

$$Max_{\{A_s\}} \ E_t\left[\sum_{s=t}^{T} \frac{U[(1+r_s)A_{s-1} + Y_s - A_s]}{(1+\delta)^{s-t}}\right].$$

- Solving for the FOC (evaluated at $s = t$):

$$U'(C_t) = E_t\left[\frac{(1+r_{t+1})U'(C_{t+1})}{1+\delta}\right].$$

This equation is the slope of the consumption in time. Notice that the MU of consumption in time t (on the left) is equal to the MU of consumption in time $t+1$ adjusted for interest (positively) and discounted by the rate of time preference (negatively). Thus, if you increase savings by one unit today (ie, consume one unit less today), then your utility today falls by U'. However, that same increase in savings today (reduction in consumption) gives you more utility from consumption tomorrow because the interest earned on savings (provided that $r > \delta$.)

- Rearranging this last equation and removing expectations (by adding an error term),

$$(1+r_{t+1})U'(C_{t+1}) = U'(C_t)(1+\delta) + \epsilon_{t+1}.$$

This is the fundamental equation that we would like to test. To render it operational, we need to make some assumptions about $U(C)$.

6.2.1 Quadratic Preferences

- Assume:

$$U(C) = aC - \frac{C^2}{2}.$$

Thus,

$$U'(C) = a - C.$$

- Our fundamental equation becomes (assuming constant interest rates):

$$(1+r)(a - C_{t+1}) = (a - C_t)(1+\delta) + \epsilon_{t+1}.$$

And solving for C_{t+1},

$$\begin{aligned}
-C_{t+1}(1+r) &= (a - C_t)(1+\delta) + \epsilon_{t+1} - a(1+r). \\
-C_{t+1}(1+r) &= a(1+\delta) - C_t(1+\delta) + \epsilon_{t+1} - a(1+r). \\
-C_{t+1} &= a\frac{(1+\delta)}{(1+r)} - C_t\frac{(1+\delta)}{(1+r)} + \frac{\epsilon_{t+1}}{1+r} - a. \\
-C_{t+1} &= \frac{a(1+\delta) - a(1+r)}{(1+r)} - C_t\frac{(1+\delta)}{(1+r)} + \frac{\epsilon_{t+1}}{1+r}. \\
-C_{t+1} &= \frac{a(\delta - r)}{(1+r)} - C_t\frac{(1+\delta)}{(1+r)} + \frac{\epsilon_{t+1}}{1+r}. \\
C_{t+1} &= \underbrace{\frac{a(r-\delta)}{(1+r)}}_{Constant} + \underbrace{C_t\frac{(1+\delta)}{(1+r)}}_{Consumption\ Last\ Period} - \underbrace{\frac{\epsilon_{t+1}}{1+r}}_{Disturbance\ Term}.
\end{aligned}$$

- Hence consumption is an $AR(1)$ process with the error term that is orthogonal to all lagged information. So when we run a regression of C_{t+1} on a constant, C_t, and lagged income variables, and we find that C_t is very significant but in some studies, the lagged income variables are as well (which goes against our model in which consumption tomorrow only depends on consumption today). There is also the possiblity that things like lagged interest rates and inflation come into play. We'll address this later.

6.2.2 Iso-Elastic Preferences - CRRA

- Assume:

$$U(C) = \frac{C^{1-\gamma} - 1}{1 - \gamma}.$$

Thus,

$$U'(C) = C^{-\gamma}.$$

- Our fundamental equation becomes:

$$C_{t+1}^{-\gamma} = \underbrace{C_t^{-\gamma}\frac{(1+\delta)}{(1+r)}}_{Consumption\ Last\ Period} - \underbrace{\frac{\epsilon_{t+1}}{1+r}}_{Disturbance\ Term}.$$

Here we get similar results to those obtained in the quadratic preferences case when we run an $AR(1)$ regression.

6.2.3 Interpreting the Error term

- In its most general form, consumption tomorrow is defined as follows:

$$C_{t+1} = K_0 + K_1 E_{t+1} W_{t+1}.$$

Hence,

$$C_{t+1} - E_t[C_{t+1}] = K_1(\underbrace{E_{t+1}W_{t+1} - E_t[W_{t+1}]}_{\text{“News” about Total Wealth}}) = \epsilon_{t+1}.$$

So the error term is basically how much we are suprised by our wealth in the current period compared with what we expected it to be last period in expectation.

- So far we have been assuming no credit restrictions, but that is not really a feasible assumption. More realistically it is probably the case that there is some segment of the population, call it λ, who are poor and don't have perfect capital markets available to them. The other segment, $(1-\lambda)$, are rich and probably can be modeled under perfect credit markets. Via some complicated mathematical techniques we can estimate the equation:

$$\Delta ln(C_{t+1}) = \alpha' + \lambda \Delta ln(Y_{t+1}) + (1-\lambda)\frac{1}{\gamma} r_{t+1} + \epsilon'_{t+1}.$$

Thus, it is clear that the poor segment bases their consumption decision only on labor income while the rich can look to the credit markets and borrow if necessary.

- If we run a regression on the variables above to estimate both γ and λ simultaneously, we find there is some evidence for the "excess sensitivity" of consumption to anticipated income changes.

- It is important to also note: "A consumer may not be currently constrainted, but the expectation of future binding credit constraints will lead him to reduce his expenditure today." All this means that the impact of credit constraints on consumption is messy and in general the level of consumption will depend on the entire income profile and not just its present value.

- Finally, if we assume perfect credit RESTRICTIONS, ie, NO BORROWING is possible, then the second budget constraint can be written:

$$A_s \geq 0 \ \ \forall s.$$

  The result is that consumption grows faster than would otherwise be the case. For a given value of lifetime wealth, this implies that the level of consumption today is DEPRESSED compared to what one would obtain in the absence of credit constraints. The regressions result is that consumption growth is between 2 and 4 percentage points faster for the non-wealthy. This result is not completely intuitive to me.

7 Week 7: 25 Feb - 1 Mar

7.1 The Intertemporal Capital Asset Pricing Model

- The idea that a risky asset demands a high return is not the end of the story. A risky asset need not show a risk premium.
- Consider the consumer's Maximization problem:

$$Max_{\{C_s, A_s^i\}} \; E_t\left[\sum_{s=t}^{T} \frac{U(C_s)}{(1+\delta)^{s-t}}\right].$$

Subject to:

$$\begin{cases} (2a) & \sum_{i=0}^{n} A_s^i \leq \sum_{i=0}^{n}(1+r_s^i)A_{s-1}^i + Y_s - C_s \qquad s = t, \ldots, T \\ (2b) & \sum_{i=0}^{n} A_T^i \geq 0 \end{cases} \tag{9}$$

- Note here the only difference is that we now include the superscript, i for each of the $i = 1, \ldots, n$ assets.
- Substituting C_s into the objective function:

$$Max_{\{A_s^i\}} \; E_t\left[\sum_{s=t}^{T} \frac{U\Big(\sum_{i=0}^{n}[(1+r_s^i)A_{s-1}^i - A_s^i] + Y_s\Big)}{(1+\delta)^{s-t}}\right].$$

- FOC:

$$E_t\left[\frac{U'(C_s)}{(1+\delta)^{s-t}}\right] = E_t\left[\frac{(1+r_{t+1}^i)U'(C_{s+1})}{(1+\delta)^{s+1-t}}\right].$$

Note this is very similar to the FOC we had previously. On the left, we have the MU at time s and on the right we have the MU at time $s+1$ with interest accummulated, and discounted by the rate of time preference. Note however the superscript on the interest rate. Thus this marginal utility equality must hold for ALL assets.

- Letting $s = t$, we have:

$$(1+\delta)U'(C_t) = E_t\left[(1+r_{t+1}^i)U'(C_{t+1})\right].$$

- Now recall that $Cov(x, y) = E(xy) - E(x)E(y)$, or $E(xy) = E(x)E(y) + Cov(x, y)$. Thus, applying this to the expectation on the right hand side,

$$(1+\delta)U'(C_t) = E_t[(1+r_{t+1}^i)]E_t[U'(C_{t+1})] + Cov(1+r_{t+1}^i, U'(C_{t+1})).$$

- Solving for the expected interest rate:

$$E_t[(1+r^i_{t+1})]E_t[U'(C_{t+1})] = (1+\delta)U'(C_t) - Cov(1+r^i_{t+1}, U'(C_{t+1})).$$

$$E_t[(1+r^i_{t+1})] = \frac{(1+\delta)U'(C_t)}{E_t[U'(C_{t+1})]} - \frac{Cov(1+r^i_{t+1}, U'(C_{t+1}))}{E_t[U'(C_{t+1})]}.$$

- Now consider asset $i = 0$ as a safe asset. The expected rate of return of this asset is equal to its actual rate of return. Then we can rewrite the equation as follows:

$$(1+r^0_{t+1}) = \frac{(1+\delta)U'(C_t)}{E_t[U'(C_{t+1})]}.$$

  Now, for some reason the riskless asset is completely uncorrelated with the marginal utility at time $t+1$. (Hence the Covariance term is zero.) I'm not completely clear why this is.

- To determine the risk premium on assets, we calculate the difference between the expected rate of return on an asset i, and the rate of return on a riskless asset. Thus,

$$E_t[r^i_{t+1}] - r^0_{t+1} = -\frac{Cov(1+r^i_{t+1}, U'(C_{t+1}))}{E_t[U'(C_{t+1})]}.$$

- **[G-7.1]** Hence the risk premium is positive if the covariance term is negative. The Covariance term is negative if the rate of return is high when the marginal utility of consumption is low. Because we are assuming strictly concave utility functions, marginal utility is low when consumption is high. When consumption is high, then a consumer would have to be given a positive risk premium for him to buy the risky asset. The idea is that he really doesn't need the high return at this point because his utility is already high.

- The Risk premium is negative if the covariance term is positive. This means that we have high rates of return when the MU is high or when consumption is therefore low. Here, the difference is actually better named a risk discount because the consumer is willing to actually pay more to buy the riskier asset. Think of an insurance contract as an example of this case.

7.2 Introduction to Economic Growth

- In this section of the course, we will study the rise in per capita income over time. Why does it happen? Why does it happen more in some economies than in others?

- Consider Kaldor's Stylized Facts as some motivating ideas in this section. First some definitions:

$$N \equiv \text{Total Labor Force.}$$
$$Y \equiv \text{Total GDP or National Income.}$$
$$K \equiv \text{Total Capital Stock.}$$
$$y = Y/N \equiv \text{Per capita Output or Labor Productivity.}$$
$$k = K/N \equiv \text{Capital per Capita or Capital Productivity.}$$

$$w \equiv \text{Average National Wage Rate.}$$
$$r \equiv \text{Economy Wide Interest Rate.}$$

$$\frac{\dot{y}}{y} = \frac{d}{dt}ln(y) \equiv \text{Rate of Output Growth.}$$
$$\frac{\dot{k}}{k} = \frac{d}{dt}ln(k) \equiv \text{Rate of Capital Growth.}$$

- Kaldor's Facts.
  - 1. Growth. y and k grow consistently and at comparable rates.
  - 2. Constancies. The output/capital stock ratio, Y/K, is constant over time (since y and k are growing at the same rate). If wN represents the economy wide wage bill, then $\frac{wN}{Y}$, or labor's share of total income, is constant over time. If rK is the economy wide capital bill, then $\frac{rK}{Y}$, or capital's share of total income, is also constant over time. Usually $\frac{wN}{Y} \simeq \frac{2}{3}$ and $\frac{rK}{Y} \simeq \frac{1}{3}$.
  - 3. Disparities. Labor productivity, $(Y/N)_j$, differs across countries, $j = 1, 2, \ldots$.

7.3 The Solow Growth Model

- Consider a production function:

$$Y = F(K, N, A),$$

and assume Y exhibits constant returns to scale (CRS) in K and N. Note K is the total capital stock, N the labor force, and A is the state of technology or otherwise called "Total Factor Productivity" or TFP. Sometime people say that Y is homogenous of degree 1 instead of CRS. We assume that Y takes on the following more specific functional form:

$$Y = F(K, NA).$$

Here Y is "Labor Augmenting" because the technological factor is multiplicative on the labor force term.

- Dividing through by NA,

$$\frac{Y}{NA} = F(\frac{K}{NA}, \frac{NA}{NA}).$$
$$\frac{y}{A} = F(\frac{k}{A}, 1).$$
$$y = A \cdot f(\frac{k}{A}).$$

  So y is known as the "labor intensive production function" or the "neoclassical production function." Note that if $\frac{k}{A}$ is about constant, than y changes only with TFP.

- A note on TFP versus y. Sometimes the literature refers to just "technology" and they might be referring to either y or A. It just depends on the context.

- Euler's Theorem. Let $Y = F(K, N)$ and F is homogeneous of degree 1. Then

$$Y = \underbrace{K\frac{\partial F}{\partial K}}_{Capital's\ Share} + \underbrace{N\frac{\partial F}{\partial N}}_{Labor's\ Share}.$$

  Thus Y, total production, is the sum of capital's share and labor's share of national income. Note also that this theorem holds even if the function is non-linear.

- Here we show Euler's theorem works for a simple cobb-douglas production function.

$$Y = N^{\alpha}K^{1-\alpha}.$$

  Via Euler's:

$$Y = K\frac{\partial F}{\partial K} + N\frac{\partial F}{\partial N}.$$
$$Y = K(1-\alpha)N^{\alpha}K^{-\alpha} + N\alpha N^{\alpha-1}K^{1-\alpha}.$$
$$Y = (1-\alpha)N^{\alpha}K^{1-\alpha} + \alpha N^{\alpha}K^{1-\alpha}.$$
$$Y = N^{\alpha}K^{1-\alpha}.$$

- So back to the production function,

$$y = A \cdot f(\frac{k}{A}).$$

  Assume that $f' > 0$ and $f'' < 0$. Thus f is concave and we have diminishing returns to scale. We also have a more technical assumption:

$$\lim_{k\to\infty} \frac{f(k)}{k} = 0.$$

Thus just means that as the economy expands and the capital stock grows, $f(k)$ grows, but less than k so the ratio of $f(k)$ to k goes to zero. Again this is diminishing marginal returns to capital but we will need the more technical form later on.

- Next we define the growth the TFP and the Labor force as :

$$\frac{\dot{A}}{A} = \xi \geq 0, \quad A(0) > 0.$$

$$\frac{\dot{N}}{N} = \nu \geq 0, \quad N(0) > 0.$$

So the growth rate of technology and the growth rate of the labor force are both CONSTANT and EXOGENOUS. Hence the Solow model is under the broad class of "Exogenous Growth Models."

- Next we define the accumulation of physical capital equation:

$$\dot{K} = \tau Y - \delta K, \quad \tau \in (0,1), \quad \delta > 0.$$

Thus τ is the savings rate and δ represents the deprecation rate on capital. Hence the positive and negative effects respectively. This function is sometimes referred to as the "Solow Savings Function."

- Now we will work a little with the Solow Savings Function. Dividing through by K,

$$\frac{\dot{K}}{K} = \tau \frac{Y}{K} - \delta.$$

Dividing the top and bottom of the first term on the right by $A \cdot N$,

$$\frac{\dot{K}}{K} = \tau \frac{Y/AN}{K/AN} - \delta.$$

Rewriting,

$$\frac{\dot{K}}{K} = \tau \frac{y/A}{k/A} - \delta.$$

Substituing in for $y = A \cdot f(\frac{k}{A})$,

$$\frac{\dot{K}}{K} = \tau \frac{Af(k/A)/A}{k/A} - \delta.$$

Canceling,

$$\frac{\dot{K}}{K} = \tau \frac{f(k/A)}{k/A} - \delta.$$

Now subtracting the growth rate of technology and of labor from both sides, noting that we write them differently on either side:

$$\frac{\dot{K}}{K} - \frac{\dot{N}}{N} - \frac{\dot{A}}{A} = \tau\frac{f(k/A)}{k/A} - \delta - \xi - \nu.$$

Rewriting,

$$\frac{\dot{K}}{K} - \frac{\dot{N}}{N} - \frac{\dot{A}}{A} = \tau\frac{f(k/A)}{k/A} - (\delta + \xi + \nu).$$

- Now a slightly tricky part. Consider the expression: $\frac{\dot{K}}{K} - \frac{\dot{N}}{N}$. Note from the definition:

$$\frac{\dot{X}}{X} = \frac{d}{dt}log\ X.$$

Applying this here,

$$\frac{\dot{K}}{K} - \frac{\dot{N}}{N} = \frac{d}{dt}log\ K - \frac{d}{dt}log\ N.$$

Or,

$$= \frac{d}{dt}(log\ K - log\ N).$$

$$= \frac{d}{dt}log\ \frac{K}{N}.$$

Noting $\frac{K}{N} = k$,

$$= \frac{d}{dt}log\ k = \frac{\dot{k}}{k}.$$

- Substituting this into the equation above,

$$\frac{\dot{k}}{k} - \frac{\dot{A}}{A} = \tau\frac{f(k/A)}{k/A} - (\delta + \xi + \nu).$$

Now, applying the trick again to the left hand side,

$$\frac{d}{dt}log\ k - \frac{d}{dt}log\ A = \tau\frac{f(k/A)}{k/A} - (\delta + \xi + \nu).$$

$$\frac{d}{dt}log\ \frac{k}{A} = \tau\frac{f(k/A)}{k/A} - (\delta + \xi + \nu).$$

Or,

$$\frac{(\dot{k/A})}{(k/A)} = \tau\frac{f(k/A)}{k/A} - (\delta + \xi + \nu).$$

- So we have a dynamical 1st order difference equation. The state of our dynamic equation is $\frac{k}{A}$.
- Setting the left hand side equal to zero,

$$0 = \tau\frac{f(k/A)}{k/A} - (\delta + \xi + \nu).$$
$$\tau\frac{f(k/A)}{k/A} = \delta + \xi + \nu.$$
$$\frac{f(k/A)}{k/A} = \frac{\delta + \xi + \nu}{\tau}.$$

  So this is the equation of the stable state equilibrium. On the right we just have a constant and on the left is a function that depends on our state. **[G-8.1]** Thus, see notes for graph, we plot $\dfrac{k}{A}$ on the horizontal axis, and we plot the constant on the right of the equation as a horizontal line on the graph. Then we graph the left hand side which is a function of $\frac{k}{A}$. Where they intersect, $[\frac{k}{A}]^*$, we have stability.

- If we are to the left of $[\frac{k}{A}]^*$, then

$$\frac{f(k/A)}{k/A} > \frac{\delta + \xi + \nu}{\tau}.$$

  And therefore,

$$\frac{(\dot{k/A})}{(k/A)} > 0,$$

  which means we move to the right towards the steady state.

- If we are to the right of $[\frac{k}{A}]^*$, then

$$\frac{f(k/A)}{k/A} < \frac{\delta + \xi + \nu}{\tau}.$$

  And therefore,

$$\frac{(\dot{k/A})}{(k/A)} < 0,$$

  which means we move to the left towards the steady state.

- Thus $[\frac{k}{A}]^*$ is a stable sink.
- Some final notes on this model. It is dynamically stable. There is a unique stable growth rate. The poorer the country, the faster it grows and the richer a country, the slower it grows. Thus we have convergence of rich and poor economies.

8 Week 8: 4 Mar - 8 Mar

8.1 More on Solow Growth Model

- Note that the state variable of the Solow model, $\frac{k}{A}$, or the per capita capital stock normalized by technology is, in equilibrium, globally stable. Note that A grows at rate ξ, so when we are at $[\frac{k}{A}]^*$, the economy settles down into some trend growth rate.

- Looking at the graph **[G-8.1]**, it is clear that the levels of depreciation, population growth, and savings rate all effect the stable equilibrium. With reference to the graph, it is clear that an economy will have a higher equilibrium $\frac{k}{A}$, and therefore national income growth, when savings is HIGHER, population growth is LOWER, or when the depreciation rate is LOWER. All these things shift down the horizontal line and thus shift the equilibrium to the right.

- Now consider the production function defined earlier, evaluated at the equilibrium value of the state variable:

$$y(t) = A(t) \cdot f(\left[\frac{k}{A}\right]^*).$$

Taking logs,

$$log\ y(t) = log f(\left[\frac{k}{A}\right]^*) + log A(t).$$

Since $A(t)$ grows at a constant rate, we can write its logarithm as the initial level of technology plus all the growth that occurs up until time t. Thus,

$$log\ y(t) = \underbrace{log f(\left[\frac{k}{A}\right]^*) + A(0)}_{Constant} + t\xi.$$

Call the constant terms, Γ_0. Thus,

$$log\ y(t) = \Gamma_0 + t\xi.$$

Thus we have a linear time trend. The growth of output is a function of the initial level of technology and the rate of growth of technology, ξ.

- Consider the level term, $\Gamma_0 = log f(\left[\frac{k}{A}\right]^*) + A(0)$. Note in equilibrium,

$$\frac{(\dot{k/A})}{(k/A)} = \tau \frac{f(k/A)}{k/A} - (\delta + \xi + \nu) = 0.$$

Or,

$$\frac{f([k/A]^*)}{[k/A]^*} = \tau^{-1}(\delta + \xi + \nu).$$

$$f([k/A]^*) = ([k/A]^*)\tau^{-1}(\delta + \xi + \nu).$$

Thus,

$$log\ f([k/A]^*) = g(\tau(\delta + \xi + \nu)^{-1}) \quad with\ g' > 0.$$

[Note we have switched the ratio.]

- Here's an example which just took me an hour to work out ... Assume:

$$f(k/A) = (k/A)^\alpha.$$

Substituting this into our dynamical first difference equation:

$$\frac{(\dot{k/A})}{(k/A)} = \tau\frac{f(k/A)}{k/A} - (\delta + \xi + \nu) = 0.$$

$$\tau\frac{(k/A)^\alpha}{k/A} = (\delta + \xi + \nu).$$

$$(k/A)^{\alpha-1} = \tau^{-1}(\delta + \xi + \nu).$$

Thus, inverting:

$$(k/A)^{1-\alpha} = \tau(\delta + \xi + \nu)^{-1}.$$

Logging,

$$log\ (k/A)^{1-\alpha} = log[\tau(\delta + \xi + \nu)^{-1}].$$

$$(1-\alpha)log\ (k/A) = log[\tau(\delta + \xi + \nu)^{-1}].$$

Raise both sides to the α,

$$(1-\alpha)log\ (k/A)^\alpha = log[\tau(\delta + \xi + \nu)^{-1}]^\alpha.$$

$$(1-\alpha)log\ (k/A)^\alpha = \alpha log[\tau(\delta + \xi + \nu)^{-1}].$$

Thus finally,

$$log\ (k/A)^\alpha = \frac{\alpha}{1-\alpha}log[\tau(\delta + \xi + \nu)^{-1}].$$

So the overall growth rate assuming this form of the production function:

$$log\ y(t) = \frac{\alpha}{1-\alpha} log[\tau(\delta + \xi + \nu)^{-1}] + A(0) + t\xi.$$

- The key point in all of this is that, in equilibrium, growth behavior is exogenous. It only depends on the exogenous growth rate of technology, ξ. Hence the Solow model is an Exogenous Growth Model.

8.1.1 Convergence

- We just determined the growth rate of the economy when IN equilibrium as:

$$log\ y(t) = \Gamma_0 + t\xi.$$

- However, when the economy is to the left or right of $[k/A]^*$, the growth rate is as follows:

$$log\ y(t) = \Gamma_0 + t\xi + \underbrace{[log\ y(0) - \Gamma_0]}_{Seperation\ Term} \overbrace{e^{\lambda t}}^{Rate\ of\ Convergence}.$$

With $\lambda = \lambda(f, (\delta + \xi + \nu), \tau)$.

- So consider the graph in slide 6 of the printed notes. We have the log of output on the vertical axis and time on the horizontal. The stable growth path is plotted as a linear function intersecting the vertical axis at Γ_0 and sloping upwards at rate, ξ.
- When $y(0)$ is somewhere off the line, we get convergence back to the stable line as shown in the graph. The further away from the stable path you start, the faster you move towards it.
- Thus we get convergence of rich and poor countries and inequality should be declining if the Solow model is really a good model of economic growth.
- In a graph of G7 GDP per worker rates from 1955 to 1990, we DO see convergence as in the Solow model. However, when we broaden the sample, we lose robustness. See graph in notes. [**G-8.2**]
- The key point it seems is that countries DO NOT have access to the same levels of technology as is assumed in Solow. Restrictions like patent laws and intellectual property rights are just a couple barriers to the flow of information and technology. The focus in the endogenous models will turn to the growth rate and level of A.

8.2 Ramsey-Cass-Koopmans Model

- The key idea in this model is that we relax the constant savings rate that we imposed on the Solow model.

- Define per capita consumption as before:

$$c(t) = \frac{C(t)}{N(t)}.$$

- Introducing a new convention: a tilde over a letter signifies that it is a per capita rate and is relative to technology. Thus,

$$\tilde{c}(t) = \frac{c(t)}{A(t)}.$$

$$\tilde{y}(t) = \frac{y(t)}{A(t)}.$$

$$\tilde{k}(t) = \frac{k(t)}{A(t)} \equiv \text{ State variable in Solow.}$$

- It is important to note that in the Solow model, $\tilde{y}$ converged to a constant.

- Define preferences as follows:

$$U(c(t)) = U(\tilde{c}(t) \cdot A(t)).$$

  And note this is a continuous time utility function. For $\rho > 0$, define the total utility level of the aggregate economy as the following:

$$\int_{t=0}^{\infty} e^{-\rho t} U(\tilde{c}(t) \cdot A(t)) \cdot N(t) dt.$$

  Noting that the growth rate of the population is constant such that:

$$\frac{\dot{N}}{N} = \nu,$$

  We can rewrite this last equation as:

$$N(0) \cdot \int_{t=0}^{\infty} e^{-(\rho-\nu)t} U(\tilde{c}(t) \cdot A(t)) dt.$$

- Define technology as before:

$$Y = F(K, NA) \Longrightarrow \tilde{y} = f(\tilde{k}) \Longleftrightarrow \frac{y}{A} = f(\frac{k}{A}).$$

- And the accumulation equation:

$$\dot{K} = \underbrace{Y - C}_{Savings} - \delta K \Longrightarrow \dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - \zeta\tilde{k}.$$

  Where,

$$\zeta = \delta + \nu + \xi > 0.$$

  So now $\dot{\tilde{k}}$ is the per capita, technology adjusted, capital stock version after accounting for a non-constant savings rate.

- From the first and second Welfare theorems, we know that with NO EXTERNALITIES, the competitive equilibrium coincides with the Social Planner's Optimization. Thus the problem is:

$$Max_{\{\tilde{c},\tilde{k}\}} \int_{t=0}^{\infty} e^{-(\rho-\nu)t} U(\tilde{c}(t) \cdot A(t)) dt.$$

  Subject to:

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - \zeta\tilde{k}.$$

- We cannot do the usual Lagrangian because of the continuous time choices. Thus we introduce a new method called the Hamiltonian. For $\lambda = \{\lambda(t) : t \geq 0\}$, define:

$$\mathbb{H}(\tilde{c}, \tilde{k}, \lambda; t) = e^{-(\rho-\nu)t} U(\tilde{c}(t) \cdot A(t)) + \left[f(\tilde{k}) - \tilde{c} - \zeta\tilde{k} \right] \lambda(t) e^{-(\rho-\nu)t}.$$

  Two differences about the Hamiltonian, notice we dropped the intergral sign and we have only included the right hand side of the constraint instead of setting it equal to zero. The idea is that c is a jumper variable so we are going to be able to set its FOC equal to zero. But since k is a crawler variable, we cannot sets its FOC equal to zero because this would imply that it could jump. The the FOCs are in general:

$$\frac{\partial \mathbb{H}}{\partial \tilde{c}(t)} = 0.$$

$$\frac{\partial \mathbb{H}}{\partial \tilde{k}(t)} = -\frac{d}{dt}\Big[\lambda(t) e^{-(\rho-\nu)t}\Big].$$

From the first FOC:

$$\frac{\partial \mathbb{H}}{\partial \tilde{c}(t)} \Longrightarrow AU'(\tilde{c}A) = \lambda.$$

From the second FOC:

$$f'(\tilde{k})\lambda(t)e^{-(\rho-\nu)t} - \zeta\lambda(t)e^{-(\rho-\nu)t} = (\rho-\nu)\lambda(t)e^{-(\rho-\nu)t} - \dot{\lambda}e^{-(\rho-\nu)t}.$$

Dividing through by $\lambda(t)e^{-(\rho-\nu)t}$,

$$f'(\tilde{k}) - \zeta = (\rho-\nu) - \frac{\dot{\lambda}}{\lambda}.$$

And in addition to these two, we also have the constraint,

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - \zeta\tilde{k}.$$

- So now define Balanced Growth Steady State (BGSS) by $\dot{\tilde{y}} = \dot{\tilde{c}} = \dot{\tilde{k}} = 0$. Note from the first FOC involving λ, since we have A on the left hand side and A is growing in equilibrium, then the left side, λ CANNOT be constant. Thus in general, $\dot{\lambda} \neq 0$.

- There is one special case when $\dot{\lambda} = 0$. Differentiate the first FOC with respect to t:

$$\frac{d}{dt}\Bigg[A(t)U'(\tilde{c}(t)A(t)) = \lambda(t)\Bigg].$$

$$\Leftrightarrow \dot{A}U'(\tilde{c}A) + AU''(\tilde{c}A)[\dot{\tilde{c}}A + \dot{A}\tilde{c}] = \dot{\lambda}.$$

Dividing through by A,

$$\Leftrightarrow \xi U'(\tilde{c}A) + U''(\tilde{c}A)[\dot{\tilde{c}}A + \dot{A}\tilde{c}] = \frac{\dot{\lambda}}{A}.$$

Or,

$$\Leftrightarrow \xi U' + U''\dot{\tilde{c}}A + U''\dot{A}\tilde{c} = \frac{\dot{\lambda}}{A}.$$

$$\Leftrightarrow U''\dot{\tilde{c}}A + U''\dot{A}\tilde{c} = -\xi U' + \frac{\dot{\lambda}}{A}.$$

$$\Leftrightarrow U''\dot{\tilde{c}}A + U''\dot{A}\tilde{c} = -(\xi - \frac{\dot{\lambda}}{AU'})U'.$$

$$\Leftrightarrow \dot{\tilde{c}}A + \dot{A}\tilde{c} = -(\xi - \frac{\dot{\lambda}}{AU'})\frac{U'}{U''}.$$

Dividing through by $A\tilde{c}$,

$$\Leftrightarrow \frac{\dot{\tilde{c}}}{\tilde{c}} + \frac{\dot{A}}{A} = -(\xi - \frac{\dot{\lambda}}{AU'})\frac{U'}{A\tilde{c}U''}.$$

Noting that $AU' = \lambda$,

$$\Leftrightarrow \frac{\dot{\tilde{c}}}{\tilde{c}} + \xi = -(\xi - \frac{\dot{\lambda}}{\lambda})\frac{U'}{A\tilde{c}U''}.$$

And substituting in the coefficient of relative risk aversion: $R(c) = -\frac{cU''(c)}{U'(c)}$,

$$\Leftrightarrow \frac{\dot{\tilde{c}}}{\tilde{c}} + \xi = (\xi - \frac{\dot{\lambda}}{\lambda})R(A\tilde{c})^{-1}.$$

Under BGSS, $\dot{\tilde{c}} = 0$, so,

$$\Leftrightarrow \xi = (\xi - \frac{\dot{\lambda}}{\lambda})R(A\tilde{c})^{-1}.$$

$$\Leftrightarrow \xi - \xi R(A\tilde{c})^{-1} = -\frac{\dot{\lambda}}{\lambda}R(A\tilde{c})^{-1}.$$

$$\Leftrightarrow \xi R(A\tilde{c}) - \xi = -\frac{\dot{\lambda}}{\lambda}.$$

And finally,

$$\Leftrightarrow \dot{\lambda} = \lambda(\xi - \xi R(A\tilde{c})).$$

SO WHEN IS $\dot{\lambda} = 0$?? Only when $R(A\tilde{c})) = 1$. Thus $U(c)$ must be of the form $U(c) = log(c)$.

- THUS, BGSS gives $\frac{\dot{y}}{y} = \frac{\dot{k}}{k} = \frac{\dot{c}}{c} = \xi = \frac{\dot{A}}{A}$. So again in equilibrium, everything grows at the rate of technological growth. Thus Ramsey is another Exogenous Growth Model.
- To study the dynamics, assume that $R(\tilde{c}A)$ is constant and take the state variables to be $(\tilde{c}, \tilde{k})$. We already have an equation for $\dot{\tilde{k}}$:

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - \zeta\tilde{k}.$$

- Our other dynamical difference equation is found by taking the second FOC and substituting in (via $\frac{\dot{\lambda}}{\lambda}$) from the differential of the first.
  So the second FOC:

$$f'(\tilde{k}) - \zeta = (\rho - \nu) - \frac{\dot{\lambda}}{\lambda}.$$

  Solving for $\frac{\dot{\lambda}}{\lambda}$,

$$\frac{\dot{\lambda}}{\lambda} = (\rho - \nu) - f'(\tilde{k}) + \zeta.$$

  Substituting this into the differential derived in the "special case" above (and letting R be a constant):

$$\frac{\dot{\tilde{c}}}{\tilde{c}} + \xi = (\xi - \frac{\dot{\lambda}}{\lambda})R(A\tilde{c})^{-1}.$$

$$\frac{\dot{\tilde{c}}}{\tilde{c}} + \xi = (\xi - (\rho - \nu) + f'(\tilde{k}) - \zeta)R^{-1}.$$

  Substituting in for $\zeta = \delta + \nu + \xi$,

$$\frac{\dot{\tilde{c}}}{\tilde{c}} + \xi = (\xi - \rho + \nu + f'(\tilde{k}) - \delta - \nu - \xi)R^{-1}.$$

$$\frac{\dot{\tilde{c}}}{\tilde{c}} + \xi = (-\rho + f'(\tilde{k}) - \delta)R^{-1}.$$

  Or finally,

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = (f'(\tilde{k}) - \delta - \rho - \xi R)R^{-1}.$$

- [**G-9.1**] See plot in notes to be discussed next week which shows what happens when we set each of the dynamic equations equal to zero and then determine the phase diagrams. We get a saddle path stable system with an upward sloping stable line. There is a unique steady state equilibrium as shown in the notes.

9 Week 9: 11 Mar - 15 Mar

9.1 More on Ramsey

- Recall the dynamic equations for the Ramsey model:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = (f'(\tilde{k}) - \delta - \rho - \xi R)R^{-1}.$$

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - \zeta\tilde{k}.$$

- See graph in notes for a plot of these equations in steady state. **[G-9.1]** Note that when $\dot{\tilde{c}} = 0$, $f'(\tilde{k}) = \delta + \rho + R\xi$, or the Marginal product of capital is constant in equilibrium.

- Note that when $\dot{\tilde{k}} = 0$, $\tilde{c} = f(\tilde{k}) - \zeta\tilde{k}$. Thus consumption equals output less the depreciation of capital. Without the ζ term, we would just have an increasing concave production function but now it is slightly twisted clockwise downward due to this extra term.

- A note on the dynamics of the system. To the left of the $\dot{\tilde{c}} = 0$ line, $\tilde{k}$ is smaller. Thus since $f'(\tilde{k}) > 0$ and $f''(\tilde{k}) < 0$, (as usual), $f'(\tilde{k})$ is larger when $\tilde{k}$ is smaller. Thus $\dot{\tilde{c}} > 0$ as shown on graph. A similar analysis can be done for the regions above and below the $\dot{\tilde{k}} = 0$ line.

- We see from the dynamics on the graph that the equilibrium is saddle path stable. Capital stock is a crawler variable and consumption is a jumper. Thus along the stable arm, capital stock converges slowly to its long run steady state. Thus income does the same since it is proportional to capital. The rate of growth of both capital and income is the same as the in the Solow model: it is equal to the rate of change of technology.

- Some empirical data. Slide 7: We see that in the past 40 years, world per capita GDP has been growing at a rate of 2.25 percent. This is unprecedented growth by historical standards. However as shown in slide 8 and 9, the poor are getting poorer and the rich are getting richer. Thus the extremes of distribution are widening. However, as shown by the percentile differentials, the economies in the middle are moving towards one end or the other as well. Comparing the richest 10 percent to the poorest 10 percent, the income differential was 12 times in 1960, whilst that differential widened to 21 times by 1990. This goes against the Solow conclusion of world wide income convergence. In slide 11, we see that in a study done by Solow, he calculated that the proportion of growth due to technology was a huge 87 percent. Clearly this is where the action is.

- US India comparison. We have three pieces of data:

$$\frac{y_{US}}{y_{INDIA}} = 14.6.$$

$$\frac{\dot{y}}{y}(US) = 1.46\%.$$

$$\frac{\dot{y}}{y}(India) = 2.32\%.$$

At first glance this looks very much like Solow convergence. But look closer. Consider a production function of the form:

$$y = A(k/A)^{\alpha}.$$

Thus,

$$MP_k = \frac{\partial y}{\partial k} = \alpha A^{1-\alpha} k^{\alpha-1} = r.$$

Substituting in for k from the production function: $k = [\frac{y}{A^{1-\alpha}}]^{1/\alpha}$,

$$r = \alpha A^{1-\alpha} \left[\frac{y}{A^{1-\alpha}}\right]^{\frac{\alpha-1}{\alpha}}.$$

Simplifying,

$$r = \alpha A^{\frac{1}{\alpha}-1} y^{1-\frac{1}{\alpha}}.$$

Assuming that $\alpha = 0.4$, and that A is the same for both countries,

$$\frac{r_{INDIA}}{r_{US}} = \frac{0.4A^{\frac{1}{0.4}-1}y^{1-\frac{1}{0.4}}}{0.4A^{\frac{1}{0.4}-1}y^{1-\frac{1}{0.4}}} = \frac{y^{1-\frac{1}{0.4}}}{y^{1-\frac{1}{0.4}}} = \frac{y^{-1.5}}{y^{-1.5}}.$$

So plugging in the income differential between US and India,

$$\frac{r_{INDIA}}{r_{US}} = \left[\frac{y_{US}}{y_{INDIA}}\right]^{1.5} = 14.6^{1.5} = 55.8!!.$$

So the growth rate differential we see between the two countries requires that we have a interest rate differential between the two countries of 56 times! Total craziness.

- Thus we now turn to alternative models because clearly Solow and Ramsey are not getting us very far.

9.2 Human Capital Models

- Define the total quantity of human capital in an economy as H. Then as usual,

$$h = \frac{H}{N} = \text{Average rate of human capital per capita.}$$

- We can define two different types of production functions:
  - $PF0$:

$$Y = F(K, AN, H).$$

    Here the human capital component is completely seperate from labor and capital. We will show that this yields:

$$\frac{\dot{y}}{y} = \frac{\dot{A}}{A} = \xi.$$

  - $PF1$:

$$Y = F(K, hAN).$$

    Here the human capital component is embodied in people, N. We will show that this yields:

$$\frac{\dot{y}}{y} = \frac{\dot{A}}{A} = \xi \text{ IFF } h \text{ is bounded.}$$

    Otherwise if h is NOT bounded, h is also a source of growth in the model. The idea that h being bounded involves that idea that people cannot learn more and more forever and ever. However, it might be unbounded in the sense that it will contribute to economic growth if this upper bound to education and learning is so far out in the future that it is not relevant.

- March Madness starts this week.
- Define as usual:

$$\tilde{h} = \frac{H}{NA} = \frac{h}{A}.$$

- Population and technology growth rates are as usual (though written slightly differently):

$$N(t) = N(0) \cdot e^{\nu t} > 0.$$
$$A(t) = A(0) \cdot e^{\xi t} > 0.$$

9.2.1 Levels but not Growth

- Ref: Mankiew, Romer, and W.
- Define the CRS production function as PF_0:

$$Y = F(K, H, NA).$$

Therefore,

$$Y/NA = F(K/NA, H/NA, NA/NA).$$
$$\tilde{y} = F(\tilde{k}, \tilde{h}, 1).$$

Or,

$$\tilde{y} = f(\tilde{k}, \tilde{h}).$$

This is very similar to the Solow model except now we have $\tilde{h}$ in the function.

- Define physical capital and human capital accumulation equation as:

$$\dot{K} = \tau_K Y - \delta_K K.$$
$$\dot{H} = \tau_H Y - \delta_H H.$$

Subject to: $\tau_K, \delta_K, \tau_H, \delta_H \geq 0$. τ's are savings rates and δ's are depreciation rates. Also, $\tau_K + \tau_H < 1$, of course.

Thus, following a similar analysis to what we did in Solow:

$$\frac{\dot{H}}{H} = \tau_H \frac{Y}{H} - \delta_H.$$

Dividing the top and bottom of the first term on the right by $A \cdot N$,

$$\frac{\dot{H}}{H} = \tau_H \frac{Y/AN}{H/AN} - \delta_H.$$

Rewriting,

$$\frac{\dot{H}}{H} = \tau_H \frac{\tilde{y}}{\tilde{h}} - \delta_H.$$

Substituing in for $\tilde{y} = f(\tilde{k}, \tilde{h})$,

$$\frac{\dot{H}}{H} = \tau_H \frac{f(\tilde{k}, \tilde{h})}{\tilde{h}} - \delta_H.$$

Now subtracting the growth rate of technology and of population from both sides, noting that we write them differently on either side:

$$\frac{\dot{H}}{H} - \frac{\dot{N}}{N} - \frac{\dot{A}}{A} = \tau_H \frac{f(\tilde{k}, \tilde{h})}{\tilde{h}} - \delta_H - \xi - \nu.$$

Rewriting,

$$\frac{\dot{H}}{H} - \frac{\dot{N}}{N} - \frac{\dot{A}}{A} = \tau_H \frac{f(\tilde{k}, \tilde{h})}{\tilde{h}} - (\delta_H + \xi + \nu).$$

- Using the trick developed in the Solow model, we can rewrite the left hand side:

$$\frac{\dot{\tilde{h}}}{\tilde{h}} = \tau_H \frac{f(\tilde{k}, \tilde{h})}{\tilde{h}} - \zeta_H$$

  Where,

$$\zeta_H = \delta_H + \xi + \nu > 0.$$

- By exactly the same method, we have:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \tau_K \frac{f(\tilde{k}, \tilde{h})}{\tilde{k}} - \zeta_K$$

  Where,

$$\zeta_K = \delta_K + \xi + \nu > 0.$$

- Thus in equilibrium,

$$\frac{\dot{\tilde{h}}}{\tilde{h}} = \frac{\dot{\tilde{k}}}{\tilde{k}} = 0.$$

  So,

$$\frac{f(\tilde{k}, \tilde{h})}{\tilde{h}} = \tau_H^{-1} \zeta_H$$

  And,

$$\frac{f(\tilde{k}, \tilde{h})}{\tilde{k}} = \tau_K^{-1} \zeta_K$$

  Which is very similar to the equilibrium level condition in the Solow model. We can't draw this one however as we now have 3 dimensions. Since F is CRS (homogenous of degree 1), then f is homogenous of degree 0. Thus, in equilbrium, we have a unique globally stable solution.

- Principal Conclusion: including human capital in this model only contributes to different levels of income but not different growth rates.

- Our conjecture is that since under Balanced Growth Steady State, k/y and h/y are constant and y is growing at some constant rate, then k and h must be growing at the same rate. This turns out to be true here, though I don't see how we proved it exactly.

- We get similar results using PF_1 and keeping h bounded. PF_1 is an IRS production function and therefore f is CRS. If f is CRS, the jacobian matrix is singular and we don't have a unique solution. We do not get convergence because increasing k, h, etc will always lead to higher levels of income.

9.2.2 Growth from Human Capital

- Take PF_1 and don't put a bound on h. h is determined endogenously. The principal conclusion is that we get both h and A becoming the engine of growth in the economy. The Average product of h and k no longer decends to zero without bound.

- H and K are very different factors of production. In H, averages are NOT important but the overall disribution of H is what really drives economic growth (or doesn't).

- Lucas 88.

10 Week 10: 18 Mar - 22 Mar

10.1 Technology And Human Capital

- Since human capital is NOT traded as is physical capital, the distribution of human capital is more important than the level of human capital.
- Technology is the accumulation of Knowledge. Knowledge can be thought of as sort of a "public good."
- Public goods have the following two properties:
  - 1) Non-Excludable: One person cannot stop anyone else from using the good.
  - 2) Non-Rival: One person's use of the good does not detract from another's use of the good.
- The obvious examples are:

$$\text{Non-excludable \& Non-rival goods} \longrightarrow \text{Sunsets.}$$

$$\text{Non-excludable \& Rival goods} \longrightarrow \text{Parking Spaces.}$$

$$\text{Excludable \& Non-rival goods} \longrightarrow \text{Computer Software.}$$

$$\text{Excludable \& Rival goods} \longrightarrow \text{Cookies.}$$

- Note that Rivalry is a physical characteristic and Excludability is a social institution of property rights.
- In today's world, knowledge isn't exactly a public good since because of intellectual property rights, it is excludable. Note that the marginal cost of disseminating knowledge to one more person (the Marginal Cost of Knowledge) is zero. This implies that in a competitive marketplace, the price of knowledge should also be zero.
- If we go for the socially efficient outcome in which we set the price of knowledge equal to zero, then this eliminates the incentive for innovation. This is really a time-inconsistency problem. We need ex-ante incentives to generate innovation. Thus the trade off between social efficiency and incentives for innovation causes a market failure.
- If economic growth is driven by technology and technology is a product of human capital/knowledge, then without intellectual property rights, there might not be enough incentive to generate innovation and thus impedes growth. If there are intellectual property rights, this creates a social inefficiency which also impedes growth. So it's a trade off which has turned into a large body of recent literature.

10.2 Endogenous Technical Change

- Romer model.
- Define ω as an "idea" and let A be the "latest idea." Thus,

$$\frac{\dot{A}}{A} = \xi H_A.$$

  Where ξ is the usual growth rate of A, but it is multiplied by H_A, or the acceleration of idea generation which comes from the proportion of the aggregate stock of human capital devoted to coming up with new ideas.

- As usual, we have the capital accumulation equation:

$$\dot{K} = Y - C.$$

  So we are ignoring depreciation in this model.

- Assume it takes η units of K and one ω (idea) to generate 1 unit of durable intermediate input, $X(\omega)$. Thus we have a lower bound for K:

$$K = \eta \int_0^\infty X(\omega) d\omega = \eta \int_0^A X(\omega) d\omega.$$

  Note that the upper limit of the integral is just A because $X(\omega) = 0$ for all $\omega > A$.

- The $X(\omega)$ function is shown in the notes. **[G-10.1]** It might be increasing which means that the more recent innovations are used more intensively than older ideas. It might be decreasing, which implies that older traditional methods are used more intensively, or finally it might be a constant function which implies we have average implementation.

- Define the production function as follows:

$$Y = g(H_Y, N) \cdot \int_0^\infty X(\omega)^\gamma d\omega \quad with\ \gamma \in (0, 1).$$

  We can think of the g function as the unskilled work force where H_Y is the proportion of the total stock of human capital devoted to manufacturing. We also assume that g is homogenous of degree $1 - \gamma$. If γ is the elasticity of substitution then the overall production function is homogenous of degree 1 or CRS. Note that if $\gamma = 1$, then the ideas that generate X are perfectly substitutable. This would imply that we could substitute the technology used to invent typewriters for the technology required to map the human genome. Hence, it makes sense that $\gamma < 1$.

- We have defined the two contributions of human capital, Y (manufacturing) and A (idea generation). Thus,

$$H = H_A + H_Y.$$

- Assume (WLOG):

$$\dot{H} = \dot{N} = 0.$$

- The decision for the unskilled agent is based on:

$$\frac{\partial Y}{\partial H_Y} \Rightarrow MP = w = g_1 \cdot \int_0^A X(\omega)^\gamma d\omega.$$

So here we have the wage of the unskilled (manufacturer) equal to his marginal product which is just the derivative of the production function with respect to H_Y. We also have the decision for the skilled (knowledge producing) agent which is based on:

$$\dot{A}P_A = \frac{\dot{A}}{A}AP_A = \xi AP_A.$$

Where $\frac{\dot{A}}{A} = \xi H_A$, or the flow of new ideas is equal to the growth rate of A times the proportion of human capital devoted to knowledge generation. Note that before $\dot{A}/A$ was just equal to ξ, but now since technology is endogenous, we multiply it by H_A.

- P_A is the price of an idea, or in our setting, the "price of a patent." Note that if knowledge was non-rival, $P_A = 0$, but we assume some degree of intellectual property rights. Define P_A as the following:

$$P_A(\omega) = \int_{t \geq 0} e^{-rt} sup_{X_t(\omega)} \left[\underbrace{g \cdot \gamma X_t(\omega)^\gamma - r\eta X_t(\omega)}_{Monopoly\ Profits} \right] dt.$$

Note this is a forward looking integral where the first term in the brackets is the benefit of an idea or the total revenue that accrues to the factory owner at scale $X_t(\omega)$. The second term in the brackets is the flow cost of running a factory at scale $X_t(\omega)$. r is a constant interest rate, η is the cost of the capital that must be "wrapped" around the idea. Note also that we are discounting this flow by the first term and we maximize (the superimum) over $X_t(\omega)$ or the scale of the factory in which we choose to exploit our idea.

- Maximizing with respect to $X_t(\omega)$ yields:

$$\gamma^2 g X_t(\omega)^{\gamma-1} - r\eta = 0.$$

Thus,

$$\gamma g X_t(\omega)^{\gamma-1} = \frac{r\eta}{\gamma} = P_X.$$

Where P_X is the "Monopoly Markup." Note that as $\gamma \longrightarrow 0$, or ideas become less and less substitutable, the markup, or degree of monopoly power, rises. Thus, $P_X \cdot X = g \cdot \gamma X_t(\omega)^\gamma$, or the total revenue generated from the idea in a factory of scale $X_t(\omega)$.

- Substituting the FOC into the P_A function yields,

$$g \cdot \gamma X_t(\omega)^\gamma - r\eta X_t(\omega) = g \cdot \gamma X_t(\omega)^\gamma - \gamma^2 g X_t(\omega)^{\gamma-1} X_t(\omega).$$

$$= g\gamma X_t^\gamma - \gamma^2 g X_t^\gamma.$$
$$= g\gamma X_t^\gamma(1-\gamma).$$
$$= \underbrace{g\gamma X_t^{\gamma-1}}_{P_X} X_t(1-\gamma).$$
$$= P_X X_t(\omega)(1-\gamma) > 0.$$

So monopoly profits are positive.

- In equilibrium, we assume symmetry and time invariance. So all ideas are implemented equally over time. Meaning typewriters are used as intensively as computers (unlikely assumption). We find:

$$w = g_1 A\bar{X}^\gamma = P_A \xi A.$$

- Or the wages equal the marginal products of both the skilled and unskilled. The implication is that that if too much of the available human capital is devoted to Knowledge building, this raises the marginal product of the manufacturing sector and draws more and more scientists to that sector away from research and development. This then raises the marginal product of working in the skilled area. So in the end we have an interior solution where wages equalize.

- An interesting thing about knowledge as an economic idea is that it is partially excludable (though still non-rival). No one owns the frontier of knowledge, but via patents and intellectual property rights, individual ideas are "owned" by a monopolist. This is a clear Bill Gates argument. Without monopoly rents generated from new ideas, there is no incentive for people to spend time learning and being innovative. If the "market" for knowledge was competitive, a zero price on an idea would eliminate the incentives required for people to innovate. However, as the monopolist grows stronger, there is another negative effect: the social loss that results from any monopoly. Thus we have a trade off between ex-ante incentives and ex-post social efficiency.

- The results of the endogenous growth model is that Y is CRS in K, N, and H_Y, A is labor augmenting, and we have diminishing returns to K.
- Real Business Cycles. See lecture notes.