Mathematics Study Guide

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September 28, 2001

1 Arithmetic of N-Tuples

- Vectors specificed by direction and length. The length of a vector is called its magnitude or "norm." For example, $x = (x_1, x_2)$. Thus, the norm of x is: $||x|| = \sqrt{x_1^2 + x_2^2}$.
- Generally for a vector, $\vec{x} = (x_1, x_2, x_3, ..., x_n), ||x|| = \sqrt{\sum_{i=1}^{n} x_i^2}$.
- Vector Order: consider two vectors, $\vec{x}, \vec{y}$. Then,

$$\vec{x} > \vec{y} \text{ iff } x_i \geq y_i \; \forall \; i \text{ and } x_i > y_i \text{ for some } i.$$

$$\vec{x} >> \vec{y} \text{ iff } x_i > y_i \; \forall \; i.$$

- Convex Sets: A set is convex if whenever is contains x' and x'', it also contains the line segment, $(1 - \alpha)x' + \alpha x''$.

2 Vector Space Formulations in Economics

- We expect a consumer to have a complete (preference) order over all consumption bundles x in his consumption set. If he prefers x to x', we write, $x \succ x'$. If he's indifferent between x and x', we write, $x \sim x'$. Finally, if he weakly prefers x to x', we write $x \succeq x'$.

- The set X, $\{x \in X : x \succeq \hat{x} \; \forall \; \hat{x} \in X\}$, is a convex set. It is all bundles of goods that make the consumer at least as well off as with his current bundle.

3 Complex Numbers

- Define complex numbers as ordered pairs such that the first element in the vector is the real part of the number and the second is complex. Thus, the real number -1 is denoted by (-1,0). A complex number, 2+3i, can be expressed (2,3).
- Define multiplication on the complex numbers as,
$$Z \cdot Z' = (a, b) \cdot (c, d) = (ac - bd, ad + bc).$$
- Let i = (0,1). Thus,
$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0).$$
- Thus $i^2 = -1$, or, $i = \pm\sqrt{-1}$.
- "i" can also be thought of as a rotation of the number line, $\frac{\pi}{2}$ anti-clockwise. Thus $-a = ai^2$
- $\sqrt{-36} = \pm 6i$.
- Addition of Complex numbers,
$$(a + bi) + (c + di) = ((a + c) + i(b + d)).$$
- Multiplication of complex numbers,
$$(a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + i(ad + bc).$$
- Define: The complex conjugate: if $Z = a + ib$, the complex conjugate of $Z, \bar{Z} = a - ib$.
- Example: $(x - iy) \cdot (x + iy) = Z\bar{Z} = x^2 + y^2$.
- Division of complex numbers. Multiply the top and bottom of the fraction by the complex conjugate of denominator which will provide a real divisor. Then divide each part of the numerator by the real denominator.
- For polynomials with real coefficients, if complex Z is a solution, so it $\bar{Z}$.
- Other complex facts:
$$Z = \bar{Z} \text{ iff } Z \text{ is Real.}$$
$$\overline{Z_1 + Z_2} = \bar{Z}_1 + \bar{Z}_2.$$
$$\overline{Z_1 \cdot Z_2} = \bar{Z}_1 \cdot \bar{Z}_2.$$
- Polar Coordinates. Let $Z = x + iy$. Thus $\sqrt{Z\bar{Z}} = \sqrt{x^2 + y^2} \equiv r$. Looking at a picture of Z in the complex plane, it is easy to see that,
$$\theta = cos^{-1}\frac{x}{r}.$$
$$\theta = sin^{-1}\frac{y}{r}.$$
Thus, $x + iy = r * cos\theta + i(rsin\theta) = r(cos\theta + isin\theta)$.

- Theorem: DeMoivre's Theorem. For $z = r(cos(\theta) + isin(\theta))$,

$$Z^N = r^N(cos(N\theta) + isin(N\theta)).$$

- Exponential Representation.

$$e^Z = \sum_{i=0}^{\infty} \frac{Z^i}{i!}.$$

$$e^{i\theta} = \sum_{j=0}^{\infty} \frac{(i\theta)^j}{j!} = cos(\theta) + isin(\theta).$$

  This is Euler's Formula: $e^{t+i\theta} = e^t(cos(\theta) + isin(\theta))$.

- Thus, the complex number $Z = a + ib$, in polar coordinates we have shown to be equal to, $r(cos\theta + isin\theta)$, with $r \equiv ||Z|| = \sqrt{Z\bar{Z}} = \sqrt{a^2 + b^2}$. Thus, it can also be written in exponential notation as,

$$a + ib = r(cos\theta + isin\theta) = re^{i\theta}.$$

- Demoivre's theorm in exponential notation: if $Z = re^{i\theta}$,

$$Z^N = (re^{i\theta})^N = r^N e^{iN\theta}.$$

- Multiplication by real numbers dilates the length of the number multiplied. Multiplication by complex numbers also rotates the number in the complex plane.

4 Matrices and Determinants

- A set of vectors are linear independent iff the only α's that satisfy,
$$\sum_k \alpha(k)x(k) = \vec{0}$$
are
$$\alpha_1 = \alpha_2 = ... = \alpha_k = 0.$$

- If, given a set of vectors, any vector in a set, X, can be written as a linear combination of those vectors, the set of vectors is said to "span" the set X.

- If N is the minimum number of vectors that will span X, then X is said to have dimension, N.

- Basis Vectors: a set of linearly independent vectors that spans a set X is called a basis for X.

- Definition: Subspace.
$$Y \subseteq X \text{ if } \forall\, y', y'' \in Y, y' + y'' \in Y \text{ and } \alpha y \in Y.$$

- Every subspace must contain the origin.

- Homogeneous Linear Equation in matrix form: $\vec{A}\vec{x} = \vec{0}$.

- The kernal of A, Ker(A), is the set of vectors, $\vec{x} \in X$, such that $\vec{A}\vec{x} = \vec{0}$.

- The rank of a matrix is the number of linearly independent columns (or rows).

- Rank(A) + dimension of Ker(A) = N (The number of columns of A).

- The system $Ax = b$ has a solution iff $Rank(A) = Rank(A|B)$.

- The system $Ax = b$, with A an MxN matrix, has exactly ONE solution iff $Rank(A) = Rank(A|B) = N$.

- Consider the nonhomogeneous linear equation, $\vec{A}\vec{x} = \vec{b}$. If this equation is to have a solution, Rank (A) = Rank ($A|b$).

- Row reduce a matrix via 1) Interchanging rows. 2) Multiplying a row by a scaler. 3) Adding a multiple of one row to another.

- Reduced Echelon Form: 1) Leading 1's in nonzero rows. 2) All zero rows at the bottom of the matrix. 3) For successive nonzero rows, leading 1 in lower row is further to the right. 4) Each column with a leading 1 has zeros everywhere else.

- Always solve for the leading variables in terms of the free variales.

- A homogeneous system of equations with free variables (more unknowns than equations) has infinitely many solutions.

- To invert a matrix, A, (the long way), augment with identity matrix. Row reduce A to the identity matrix applying transformations to the identity matrix all along. The originial identity matrix becomes A^{-1}.

- Transpose Rules.

$$(A+B)^T = A^T + B^T.$$
$$(A-B)^T = A^T - B^T.$$
$$(A^T)^T = A$$
$$(rA)^T = rA^T$$
$$(AB)^T = B^T A^T.$$

- Idempotent Matrix: $\vec{A} \cdot \vec{A} = \vec{A}$.

- Permutation Matrix: a square matrix in which each row and each column contains only one 1.

- Nonsingular matrix: A matrix of full rank, Rank(A) = N = The number of columns.

- Orthogonal Matrix: A is orthogonal iff $AA^T = I = A^T A$ and therefore $A^T = A^{-1}$.

- Theorem: A square matrix can have at most one inverse.

- Theorem: If A^{-1} exists and $\vec{A}$ is nonsingular, the unique solution to $\vec{A}\vec{x} = \vec{b}$ is,

$$\vec{x} = \vec{A}^{-1}\vec{b}.$$

- Inverse properties for invertible matricies, $\vec{A}$ and $\vec{B}$,

$$(A^{-1})^{-1} = A.$$
$$(A^T)^{-1} = (A^{-1})^T.$$
$$(AB)^{-1} = B^{-1}A^{-1}.$$

- Theorem: If A is invertible,

$$A^M \text{ is invertible.}$$
$$(A^M)^{-1} = (A^{-1})^M.$$
$$A^{M_1}A^{M_2} = A^{M_1+M_2}.$$
$$(rA)^{-1} = \frac{1}{r}A^{-1}.$$

4.1 Determinants

- For a 2x2 array, $|A| = a_{11}a_{22} - a_{12}a_{21}$.
- M_{ij}, obtained by deleting the i^{th} row and j^{th} column and computing the determinant of those rows and columns remaining, is called the $(i,j)_{th}$ minor of A.
- Cofactor: $C_{ij} = (-1)^{i+j}M_{ij}$ is called the cofactor of A. A cofactor is a signed minor.
- If A has a row or column of all zeros, $|A| = 0$.
- If A is an upper or lower triangular matrix, $|A|$ is simply the product of its diagonal entries.
- $|A^T| = |A|$.
- $|AB| = |A||B|$.
- Define A an NxN matrix with C_{ij} as the cofactor of a_{ij}. Then the NxN matrix,

$$C = \begin{bmatrix} C_{11} & C_{12} & ... & C_{1n} \\ C_{21} & C_{22} & ... & C_{2n} \\ C_{31} & C_{32} & ... & C_{3n} \\ C_{41} & C_{42} & ... & C_{4n} \\ ... & ... & ... & ... \\ C_{n1} & C_{n2} & ... & C_{nn} \end{bmatrix}^T \tag{1}$$

  is called the Adjoint of A denoted adj(A).
- Definition of the Inverse of the matrix A.

$$A^{-1} = \frac{1}{|A|}adj(A).$$

- Cramer's Rule: Consider $Ax = b$, N linear equations in N unknowns, $|A| \neq 0$, and

$$b = \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix}. \tag{2}$$

  Then,

$$x_i = \frac{|A_i|}{|A|} \text{ for } i = 1...N.$$

  Where A_i is obtained by replacing the i_{th} column of A by b.

4.2 Eigenvalues and Eigenvectors

- $\vec{A}\vec{x} = \lambda\vec{x}$ where λ is the eigenvalue and $\vec{x}$ is the eigenvector.
- To find the characteristic polynomial of A, compute,

  $$det(A - \lambda I) = 0.$$

  Solve for λ which gives you the eigenvalues of the matrix.
- If λ is an eigenvalue for A, then there exists nontrivial solutions to $(A - \lambda I)x = 0$.
- To find the corresponding eigenvectors, solve, $(A - \lambda I)x = 0$, with your value of λ substituted in.
- Eigenvalues of triangular matrices are just the diagonal entries of the matrix.
- To Diagonalize a matrix.
  - 1) Find N linearly independent eigenvectors, $p_1, p_2, ..., p_n$.
  - 2) Form a matrix P with $p_1, p_2, ..., p_n$ as its column vectors.
  - 3) Compute P^{-1}.
  - 4) Then $P^{-1}AP = D$. D is a diagonal matrix with the original eigenvalues of A as its diagonal entries.
- Theorem: If $x_{1...N}$ are eigenvectors of A corresponding to distinct eigenvalues, $\lambda_{1...N}$, then $x_{1..N}$ is a linearly independent set.
- Theorem: If the NxN matrix, A, has N distinct eigenvalues, then A is diagonalizable. But the converse is not true ... A may still be diagonalizable even though it does not have N distinct eigenvalues.
- One can compute powers of a matrix using its diagonal. $A^M = PD^MP^{-1}$.
- Theorem: A matrix with a repeated eigenvalue but without corresponding independent eigenvectors is called nondiagonalizable. If A is a 2x2 with repeated eigenvalues, A is diagonalizable iff A is already diagonal.
- For nondiagonalizable matrices, use an "almost" diagonal matrix. Compute the one eigenvector as normal, $\vec{v_1}$. Compute $[\lambda_1 I - A][\vec{v_2}] = \vec{v_1}$. Augment v_1 and v_2 and this gives you your P matrix. Compute D as normal.

4.3 Inner Products, Orthogonal Bases and Orthonormal Bases

- Inner Product: $\vec{X} \cdot \vec{Y} = x_1y_1 + x_2y_2 + ... + x_Ny_N$.
- $\vec{X} \cdot \vec{Y} = ||X|| * ||Y||cos\theta$.
- If $\vec{X} \cdot \vec{Y} = 0$ then X and Y are perpendicular.

- Definition: A set of vectors is called orthogonal if all pairs of vectors in the set are orthogonal. An orthogonal set in which each vector has $norm = 1$, is called orthonormal.

- When diagonalizing a matrix, if P is orthogonal, then P is said to orthogonally diagonalize A such that $D = P^{-1}AP$. A must be symmetric for this to be possible.

- To convert a set of eigenvectors to orthonormal, use Gram-Schimidt (refer to notes).

4.4 Quadratic Forms

- Writing a quadratic equation in matrix form: $x^T Ax$.

- Example: writing $x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3$ in quadratic form:

$$y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} * \begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 3 \\ -1 & 3 & -3 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \quad (3)$$

- $x^T Ax$ is positive definite if $x^T Ax > 0$ for all $x \neq 0$. A is called a positive definite matrix if $x^T Ax$ is a positive definite quadratic form.

- A symmetric matrix, A, is positive definite iff all eigenvalues of A are positive.

- A symmetric matrix, A, is positive definite iff the determinant of every principal submatrix is positive.

- A is an NxN matrix. A kXk submatrix of A formed by deleting $N - k$ columns and the same $N - k$ rows is called a k_{th} order principal submatrix of A. The determinant of a kXk PSM is called a k_{th} order principal minor of A.

- The k_{th} order principal submatrix obtained by deleting the last $N - k$ rows and the last $N - k$ columns from a NxN matrix, A, is called the k_{th} order leading principal submatrix of A. Its determinant is called the k_{th} order leading principal minor of A.

- Theorem: A is NxN symmetric. A is positive definite iff all its N leading principal minors are strictly positive. A is negative definite iff its N leading principal minors alternate in sign so that $|A_{1^{st}}| < 0, |A_{2^{nd}}| > 0$, etc.

- Positive Semidefinite: $x^T Ax \geq 0 \; \forall \; x$.

- Negative Semidefinite: $x^T Ax \leq 0 \; \forall \; x$.

- Indefinite: $x^T Ax$ takes on both positive and negative values.

- Special properties of symmetric matrices: All complex eigenvalues are actually real, orthogonal eigenvectors, and for each eigenvalue of multiplicity m, there are m linearly independent eigenvectors which can also be orthonormalized.

4.5 Quadratic Forms with Linear Constraints

- To find if for a matrix A, NxN, $Q(x) = x^T Ax$ on $Bx = 0$ where B is MxN then form the Hessian:

$$H = \begin{bmatrix} 0 & B \\ B^T & A \end{bmatrix}. \tag{4}$$

- If $|H|$ has same sign as $(-1)^N$ and if last $N - M$ leading principal minors alternate in sign, then Q is negative definite on $Bx = 0$.

- If $|H|$ and the last $N - M$ principal minors all have the same as sign as $(-1)^M$ then Q is positive definite on $Bx = 0$.

- If either of the past two conditions fail by nonzero leading principal minors, then Q is indefinite on $Bx = 0$.

5 Differential Equations

- $x'(t) = ax$.
- Seperation of Variables technique. Rewrite as, $\frac{1}{x}dx = adt$.
- Intergrate both sides to get $lnx = at + lnc$.
- Rearrange: $x = ce^{at}$
- Integrating Factors technique.
  - Properties of Linear DE: If y_p and y_h are both solutions to a differential equation, (particular and homogeneous), then $y_p + y_h$ is also a solution.
  - Using the integrating factor to solve linear differential equations such that $\frac{dy}{dt} + P(t)y = f(t)$.
  - The integrating factor is $e^{(\int P(t)dt)}$.
  - Multiply both sides by the integrating factor.
  - $e^{(\int P(t)dt)}\frac{dy}{dt} + e^{(\int P(t)dt)}P(t)y = e^{(\int P(t)dt)}f(t)$.
  - Then via chain rule ...
  - $\frac{d\{e^{(\int P(t)dt)}y\}}{dt} = e^{(\int P(t)dt)}f(t)$.
  - Then integrate to find solution.
- A differential equation, $Ax = b$ is said to be homogeneous if $b = \theta$. The solution space, those x's that satisfy this equation, is called the ker (A), or the complementary function.
- The particular solution to $Ax = b$ is also called the particular integral.
- To solve second order differential equations. First find the auxiliary polynomial (characteristic equation) and factor. For example, the a.p. of $ax''(t) + bx'(t) + cx(t) = f(t)$ is $ax^2 + bx + c = 0$. Solving this will provide roots, λ. Then solutions will be of the form:

$$x = \gamma e^{\lambda t} \text{ for a non repeated root.}$$

$$x = \gamma_0 e^{\lambda_0 t} + \gamma_1 e^{\lambda_1 t} \text{ for two roots.}$$

$$x = \gamma_0 e^{\lambda_0 t} + \gamma_1 t e^{\lambda_1 t} \text{ for two repeated roots.}$$

$$x = \gamma_1 e^{\lambda t} cos(\mu t) + \gamma_2 e^{\lambda t} sin(\mu t) \text{ for complex roots } \lambda \pm i\mu.$$

- If a root is repeated more than once, (real or complex), simply add those terms on with higher powers of t.

5.1 Nonhomogeneous Differential Equations

- $Ax = b$.
- If A has constant coefficients, try to find a particular solution with similar form to b.
- Example: $x''(t) - 5x'(t) + 6x = e^t$.
  - Auxiliary polynomial: $s^2 - 5s + 6 = 0$. Or $(s-3)(s-2) = 0$. So roots are 2 and 3. General homogeneous solution is therefore, $X_h = \gamma_1 e^{2t} + \gamma_2 e^{3t}$. Guess at an appropriate particular solution. Since b is an exponential, guess $x_p(t) = \beta e^t$. Therefore,
  $$x_p(t) = x_p'(t) = x_p''(t) = \beta e^t.$$
  Substitute these equations into our original secord order equation to get:
  $$\beta e^t - 5\beta e^t + 6\beta e^t = e^t.$$
  Now simply equate coefficients,
  $$\beta - 5\beta + 6\beta = 1.$$
  $$\beta = \frac{1}{2}.$$
  Thus our particular solution is,
  $$X_p(t) = \frac{1}{2}e^t.$$
  And thus our general solution,
  $$X(t) = X_h(t) + X_p(t) = \gamma_1 e^{2t} + \gamma_2 e^{3t} + \frac{1}{2}e^t.$$
- If your guess at the form of the particular solution is already in the homogeneous solution, simply multiply the particular guess by t.

5.2 More on Nonhomogeneous with constant coefficents

- General solution = $y(t) = y_h + y_p$.
- Polynomial $f(t)$.
  - Look for particular solution of the form $y_p = At^n + Bt^{n-1} + Ct^{n-2} + ... + Dt + E$.
- Exponential $f(t)$.
  - Look for particular solution of the form $y_p = Ae^{pt}$.
- Sine or Cosine $f(t)$.

- Look for particular solution of the form $y_p = Asin(at) + Bcos(at)$.

- Complex $f(t)$.
  - If $b = 8e^{(3+2i)t}$, Look for particular solution of the form $y_p = (\beta_1 + i\beta_2)e^{(3+2i)t}$.
- Combination $f(t)$.
  - $f(t) = P_n(t)e^{at}, \Rightarrow y_p = (At^n + Bt^{n-1} + Ct^{n-2} + ... + Dt + E)e^{at}$.
  - $f(t) = P_n(t)sin(at)$ or $P_n(t)cos(at), \Rightarrow y_p = (A1t^n + A2t^{n-1} + A3t^{n-2} + ... + A4t + A5)cos(at) + (B1t^n + B2t^{n-1} + B3t^{n-2} + ... + B4t + B5)sin(at)$.
  - $f(t) = e^{at}sin(bt)$ or $e^{at}cos(bt), \Rightarrow y_p = Ae^{at}cos(bt) + Be^{at}sin(bt)$.
  - $f(t) = P_n(t)e^{at}sin(bt)$ or $P_n(t)e^{at}cos(bt), \Rightarrow y_p = (A1t^n + A2t^{n-1} + A3t^{n-2} + ... + A4t + A5)e^{at}cos(bt) + (B1t^n + B2t^{n-1} + B3t^{n-2} + ... + B4t + B5)e^{at}sin(bt)$.
- Superposition $f(t)$.
  - If $f(t)$ is the sum of m terms of the forms previously described.
  - $y_p = y_{p1} + y_{p2} + y_{p3} + ... + y_{pm}$.

5.3 Economics Applications of Differential Equations

- Usually the D.E.'s are too difficult to solve analytically, so we must resort to graphs.
- Draw direction field as a series of short line segments that are the slope of equation at different points. Gives an overall picture of what the equation or system looks like without ever solving for it.
- Another type of graph where you plot contour lines of all values where the slope of the equation is the same. (connecting like line segments on the direction field).
- Key to these graphs is to first find the stationary / equilibrium points and then determine the slopes inbetween these phase points. One can then draw a phase line which summarizes the behavior of a system in one vertical (or horizontal) line.

5.4 Second Order Differential Equations

- Form: $\dfrac{d^2y}{dt^2} + p(t)\dfrac{dy}{dt} = q(t)y = f(t)$.
- Homogeneous if $f(t) = 0$.
- given solutions y_1 and y_2 to the 2nd order differential equation, you must check the Wronskian if both solutions are from real roots of the characteristic.
- $$\mathbf{W} = det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}. \tag{5}$$

- If W is equal to 0 anywhere on the interval of consideration, then y_1 and y_2 are not linearly independent.
- General solution given y_1 and y_2 is found as usual by the linearization theorem.
- Characteristic polynomial of a 2nd order with constant coefficients: $as^2 + bs + c = 0$.
- Solutions of the form $y(t) = e^{st}$.
- $s = -\frac{b}{2a} + / - \frac{\sqrt{b^2 - 4ac}}{2a}$.
  - if $b^2 - 4ac > 0$, then two distinct real roots.
  - if $b^2 - 4ac < 0$, then complex roots.
  - $b^2 - 4ac = 0$, then repeated real roots.

5.5 Two real distinct Roots

- Two real roots, s_1 and s_2.
- General solution = $y(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$.

5.6 Complex Roots

- Complex Roots, $s_1 = p + iq$ and $s_2 = p - iq$.
- General solution = $y(t) = k_1 e^{pt} cos(qt) + k_2 e^{pt} sin(qt)$.

5.7 Repeated Roots

- Repeated Root, s_1.
- General solution = $y(t) = k_1 e^{-\frac{b}{2a}t} + k_2 t e^{-\frac{b}{2a}t}$.

5.8 Integration by Parts

$\int udv = uv - \int vdu$.

6 Phase Portraits for Autonomous Systems

- The solution to $Ax = 0$ is a fixed point. If A is nonsingular, then the origin is the only fixed point.
- θ is always a stationary point.
- One can use the eigenvectors to mark out the phase barriers of the system.

6.1 Straightline Solutions, Eigenvectors and Eigenvalues

- A straightline solution to the system $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$ exists provided that,

$$\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}. \tag{6}$$

- To determine λ, compute the det[($\mathbf{A}$ - λI)] =

$$det\begin{bmatrix} a-\lambda & b \\ c & e-\lambda \end{bmatrix} = (a-\lambda)(e-\lambda) - bc = 0. \tag{7}$$

.

- This expands to the characteristic polynomial =

$$\lambda^2 - (a-d)\lambda + ae - bc = 0.$$

- Solving the characteristic polynomial provides us with the eigenvalues of A.

6.2 Stability

Consider a linear 2 dimensional system with two nonzero, real, distinct eigenvalues, λ_1 and λ_2.

- If both eigenvalues are positive then the origin is a source (unstable).
- If both eigenvalues are negative then the origin is a sink (stable).
- If the eigenvalues have different signs, then the origin is a saddle (unstable).

6.3 Complex Eigenvalues

- Euler's Formula: $e^{a+ib} = e^a e^{ib} = e^a cos(b) + ie^a sin(b)$.
- Given real and complex parts of a solution, the two parts can be treated as seperate independent solutions and used in the linearization theorem to determine the general solution.

- Stability: consider a linear two dimensional system with complex eigenvalues $\lambda_1 = a+ib$ and $\lambda_2 = a - ib$.
  - If a is negative then solution spiral towards the origin (spiral sink).
  - If a is positive then the solutions spiral away from the origin (spiral source).
  - If $a = 0$ the solutions are periodic closed paths (neutral centers).

6.4 Repeated Eigenvalues

- Given the system, $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$ with one repeated eigenvalue, λ_1.
- If **V1** is an eigenvector, then $X_1(t) = e^{\lambda t}V_1$ is a straight line solution.
- Another solution is of the form $X_2(t) = e^{\lambda t}(tV_1 + V_2)$.
- Where $V_1 = (A - \lambda I)V_2$.
- X_1 and X_2 will be independent and the general solution is formed in the usual manner.

6.5 Zero as an Eigenvalue

- If zero is an eigenvalue, nothing changes but the form of the general solution is now

$$\mathbf{X}(t) = k_1\mathrm{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2.$$

6.6 Solving Non-linear Systems by a Linear Approximation

- First find stationary points, build up local phase portraits, then try to construct global portraits with a little bit of artistic imagination.
- Example: $x_1' = x_2 + x_1x_2$ and $x_2' = x_1^2 - x_2^2 - 2x_1$.
  - Set both equations equal to zero simultaneously and find stationary points at: $(-1, -\sqrt{3}), (0, 0), (2, 0), (-1, \sqrt{3})$.
  - The linear approximation to these equations near a given stationary point x* is obtained from a 1^{st} order taylor series' about x*.
  - So we can write a linear approximation at x* (Using the Jacobian matrix, A) as,

$$= \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{8}$$

So, in our example,

$$A = \begin{bmatrix} x_2 & 1 + x_1 \\ 2x_1 - 2 & -2x_2 \end{bmatrix} \tag{9}$$

– Plug in each of the four stationary points into this matrix and find the eigenvalues.

$$\text{at } (0,0) \longrightarrow \lambda = \pm\sqrt{2}i \longrightarrow \text{Neutral Center}$$

$$\text{at } (2,0) \longrightarrow \lambda = \pm\sqrt{6} \longrightarrow \text{Saddle}$$
$$\text{at } (-1,\sqrt{3}) \longrightarrow \lambda = \sqrt{3}, -2\sqrt{3} \longrightarrow \text{Saddle}$$
$$\text{at } (-1,-\sqrt{3}) \longrightarrow \lambda = -\sqrt{3}, 2\sqrt{3} \longrightarrow \text{Saddle}$$

– Plot each local portrait and try to artistically connect.

7 Multivariate Calculus and Optimisation

- Define $df = f'(x)dx$.
- Differentials are exact for $\Delta x = dx$ near zero.
- Usual basic calculus 2nd derivative test. Given x, such that, $f'(x) = 0$,

$$f''(x) < 0 \Rightarrow \text{Relative Maximum.}$$

$$f''(x) > 0 \Rightarrow \text{Relative Minimum.}$$

- Functions of several variables.

$$df = \frac{\delta f}{\delta x_1}(x^o)dx_1 + \frac{\delta f}{\delta x_2}(x^o)dx_2 + ... + \frac{\delta f}{\delta x_n}(x^o)dx_n.$$

- The total differential of a function is the sum of its partial derivatives.
- Directional Derivative. Define a direction vector, s. Thus,

$$\frac{\delta f}{\delta s} = s_1\frac{\delta f}{\delta x_1} + s_2\frac{\delta f}{\delta x_2} + ... + s_n\frac{\delta f}{\delta x_n} = Df * s.$$

- Define: Df,

$$\mathbf{Df} = \begin{bmatrix} \frac{\delta f}{\delta x_1} & \frac{\delta f}{\delta x_2} & ... & \frac{\delta f}{\delta x_n} \end{bmatrix}. \quad (10)$$

- Gradiant. Define the gradiant of f to be:

$$\bigtriangledown f \begin{bmatrix} \frac{\delta f}{\delta x_1} \\ \frac{\delta f}{\delta x_2} \\ \frac{\delta f}{\delta x_3} \\ ... \\ \frac{\delta f}{\delta x_n} \end{bmatrix}. \quad (11)$$

- Thus, $\bigtriangledown f = [Df]^T$.
- Result: $Df * s = \frac{\delta f}{\delta s} = [\bigtriangledown f]^T * s$ gives an approximation to the change in f in moving $||s||$ in the s-direction.

- Define the Hessian matrix. $H = D^2 f(x) = D[\bigtriangledown f(x)]$. Thus,

$$\mathbf{H} = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1 \delta x_1} & \frac{\delta^2 f}{\delta x_1 \delta x_2} & \cdots & \frac{\delta^2 f}{\delta x_1 \delta x_n} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2 \delta x_2} & \cdots & \frac{\delta^2 f}{\delta x_2 \delta x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta^2 f}{\delta x_n \delta x_1} & \frac{\delta^2 f}{\delta x_n \delta x_2} & \cdots & \frac{\delta^2 f}{\delta x_n \delta x_n} \end{bmatrix}. \tag{12}$$

- Generalization of 2nd order conditions for determining maxima and minima. For a local maximum, we need $\frac{\delta^2 f}{\delta s^2} < 0$ for all s. This can be written in matrix form as,

$$\frac{\delta^2 f}{\delta s^2} = s^T D^2 f(x) s.$$

- So at the top of a hill, we need $s^T D^2 f(x) s < 0$.
- In summary, for a local Maximum
  - 1) $\triangle f(x*) = (0, 0, ..., 0)$
  - 2) $det(d^2 f(x*)) > 0$
  - 3) $\frac{\delta^2 f}{\delta x_1^2} < 0$.
- In summary, for a local Minimum
  - 1) $\triangle f(x*) = (0, 0, ..., 0)$
  - 2) $det(d^2 f(x*)) > 0$
  - 3) $\frac{\delta^2 f}{\delta x_1^2} < 0$.
- If $det(d^2 f(x*)) < 0$, then critical point is a saddle point.

7.1 The Chain Rule

- If,

$$x_1 = x_1(t),$$
$$x_2 = x_2(t),$$
$$x_3 = x_3(t),$$

Then,

$$\frac{df(x_1, x_2, x_3)}{dt} = \frac{\delta f}{\delta x_1}\frac{dx_1}{dt} + \frac{\delta f}{\delta x_2}\frac{dx_2}{dt} + \frac{\delta f}{\delta x_3}\frac{dx_3}{dt}.$$

- Use a tree diagram for a simple way of showing this for any degree of complication. (See notes)
- Linear Intergrals don't make any sense. (see notes for more confusion)
- Slutsky Symmetry Conditions:
$$\frac{\delta x_i}{\delta p_j} = \frac{\delta x_j}{\delta p_i}.$$
- Definition of Concave Functions: $f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$.
- Calculus Criteria for Concave Functions: $f(x') - f(x) \leq f'(x)(x' - x)$.
- Hessian Criteria for Concave Functions: $H = D^2 f(x)$ is negative semidefinite for all x. (Iff its $2^N - 1$ principal minors alternates in signs - odd ones ≤ 0 and even ones ≥ 0.)
- Hessian Criteria for Convex Functions: $H = D^2 f(x)$ is positive semidefinite for all x. (Iff its $2^N - 1$ principal minors are ≥ 0.)

8 Consumer Theory, Kuhn Tucker, Envelopes, and IFT

- Lagrangian: Maximize $U(x_1, ..., x_n)$ s.t. $\sum p_i x_i - m \leq 0$ and $x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$.
- $L(x, \lambda) = U(x) - \lambda(px - m)$.
- First order Conditions:
  - $\frac{\delta L(x, \lambda)}{\delta x_i} = 0$ forall i.
  - $-\lambda \frac{\delta L}{\delta \lambda} = 0$.
  - $\lambda > 0$. (Binding Constraint)
  - $px - m \leq 0$ (Constraint)
  - $-x \leq 0$.
- Kuhn Tucker Formulation.
- For problems involving only inequality constraints and a complete set of nonnegativity constraints. Max $f(.)$ s.t. $g(x, \alpha) \leq 0$ and $x_i \geq 0\ \forall i$.
- Normal Lagrangian would be:

$$L(x, \lambda, \nu, \alpha) = f(x) - \lambda g(x, \alpha) + \nu_1 x_1 + \nu_2 x_2 + ... + \nu_n x_n.$$

- With Kuhn Tucker, rewrite as $\bar{L}$ where,

$$L(x, \lambda, \nu) = \bar{L}(x, \lambda) = \nu x.$$

- New First order conditions are,
  - $\frac{\delta L}{\delta x_i} = \frac{\delta \bar{L}}{\delta x_i} = 0$ forall i OR $\frac{\delta \bar{L}}{\delta x_i} = -\nu_i$.
  - $\frac{\delta \bar{L}}{\delta x_i} \leq 0$ AND $x_i \frac{\delta \bar{L}}{\delta x_i} = 0$.
  - AND $\frac{-\delta \bar{L}}{\delta \lambda} = \frac{-\delta L}{\delta \lambda} = g(x^*(\alpha)) \leq 0$.
- However, Kuhn Tucker is really just a special case so it's better to stick with the original formulation.

8.1 Envelope Theorem

- max $f(.\alpha)$ s.t. $g(.,\alpha) \leq 0$ and $x \geq 0$.
- Define Maximum Value Function:

$$F(\alpha) = f(x^*(\alpha), \alpha) \text{ such that } f(x^*, \alpha) \text{ is a maximum.}$$

- Theorem: $\bigtriangledown_\alpha F = \frac{\delta L}{\delta \alpha}(x^*, \lambda^*, \alpha)$.
- Consider $L \equiv U(x) - \lambda(px - m)$. Then,

$$\frac{\delta L}{\delta m} = \frac{\delta V}{\delta m} = \lambda^*(p, m).$$

$$\frac{\delta L}{\delta p} = \frac{\delta V}{\delta p} = -\lambda^*(p, m)x_i^*(p, m).$$

Thus,

$$-\frac{\frac{\delta V}{\delta p}}{\frac{\delta V}{\delta m}} = x_i^*$$

.

8.2 Linear Implicit Function Theorem

- It is common in economics when one has a theory or relationship involving a number of variables to consider the dependent relationship of some of the variables.
- Theorem: If the relationship betwen the variables in linear, one can write, $Ax = b$. Let $x_1...x_N$ and $x_{N+1} = b_1, x_{N+2} = b_2...x_{N+M} = b_M$ be a partition of $N + M$ variables into endogenous and exogenous variables respectively.
- There is, for each choice of b, a unique set of values $(x_1^o, x_2^o, ..., x_N^o)$ Iff a) The number of endogenous variables equals the number of equations (N); b)the Rank of A where $Ax = b$ is N (Full Rank).
- Consider a function, $G(x^*, y^*) = 0$ and $\frac{\delta G}{\delta y} \neq 0$. Thus, in the neighborhood of (x^*, y^*),

$$\frac{dy}{dx_i}(x^*) = \frac{\frac{-\delta G}{\delta x_i}(x^*, y^*)}{\frac{\delta G}{\delta y}(x^*, y^*)}.$$

- Done.