Differential Equations Study Sheet

Matthew Chesnes
It’s all about the Mathematics!
Kenyon College

Exam date: May 11, 2000
6:30 P.M.
1 First Order Differential Equations

- Differential equations can be used to explain and predict new facts for about everything that changes continuously.

$$\frac{d^2x}{dt^2} + a \frac{dx}{dt} + kx = 0.$$  

- $t$ is the independent variable, $x$ is the dependent variable, $a$ and $k$ are parameters.

- The order of a differential equation is the highest derivative in the equation.

- A differential equation is linear if it is linear in parameters such that the coefficients on each derivative of $y$ term is a function of the independent variable ($t$).

- Solutions: Explicit $\rightarrow$ Written as a function of the independent variable. Implicit $\rightarrow$ Written as a function of both $y$ and $t$. (defines one or more explicit solutions.

1.1 Population Model

- Model: $\frac{dP}{dt} = kP$.

- Equilibrium solution occurs when $\frac{dP}{dt} = 0$.

- Solution: $P(t) = Ae^{kt}$.

- If $k > 0$, then $\lim_{t \to \infty} P(t) = \infty$. If $k < 0$, then $\lim_{t \to \infty} P(t) = 0$.

- Redefine model so it doesn’t blow up to infinity.

$$\frac{dP}{dt} = kP(1 - \frac{P}{N}).$$

- $N$ is the carrying capacity of the population.

1.2 Separation of Variables Technique

- $\frac{dy}{dt} = g(t)h(y)$.

- $\frac{1}{h(y)}dy = g(t)dt$.

- Integrate both sides and solve for $y$.

- You might lose the solution $h(y) = 0$. 
1.3 Mixing Problems

- \( \frac{dQ}{dt} = \text{Rate In} - \text{Rate Out}. \)
- Consider a tank that initially contains 50 gallons of pure water. A salt solution containing 2 pounds of salt per gallon of water is poured into the tank at a rate of 3 gal/min. The solution leaves the tank also at 3 gal/min.
- Therefore Input = \(2 \text{(lb/gal)} \times 3 \text{(gal/min)}.\)
- Output = \(? \text{(lbs/gal)} \times 3 \text{(gal/min)}.\)
- Salt in Tank = \(\frac{Q(t)}{50}.\)
- Therefore output of salt = \(\frac{Q(t)}{50} \text{(lbs/gal)} \times 3 \text{(gal/min)}.\)
- \(\frac{dQ}{dt} = \text{Rate In} - \text{Rate Out} = 2 \text{ lbs/gal} \times 3 \text{gal/min} - \frac{Q(t)}{50} \text{(lbs/gal)} \times 3 \text{(gal/min)}.\)
- 6 lbs/min - \(\frac{3Q(t)}{50}\) lbs/min.
- Solve via separation of Variables.

1.4 Existance and Uniqueness

- Given \( \frac{dy}{dt} = f(t, y). \) If \( f \) is continuous on some interval, then there exists at least one solution on that interval.
- If both \( f(t, y) \) and \( \frac{\partial}{\partial y} f(t, y) \) are continuous on some interval then an initial value problem on that interval is guaranteed to have exactly one Unique solution.

1.5 Phase Lines

- Takes all the information from a slope fields and captures it in a single vertical line.
- Draw a vertical line, label the equilibrium points, determine if the slope of \( y \) is positive or negative between each equilibrium and label up or down arrows.

1.6 Classifying Equilibria and the Linearization Theorem

- Source: solutions tend away from an equilibrium \( \rightarrow f'(y_o) > 0. \)
- Sink: solutions tend toward an equilibrium \( \rightarrow f'(y_o) < 0. \)
- Node: Neither a source or a sink \( \rightarrow f'(y_o) = 0 \) or DNE.
1.7 Bifurcations

- Bifurcations occur at parameters where the equilibrium profile changes.
- Draw phase lines ($y$) for several values of $a$.

1.8 Linear Differential Equations and Integrating Factors

- Properties of Linear DE: If $y_p$ and $y_h$ are both solutions to a differential equation, (particular and homogeneous), then $y_p + y_h$ is also a solution.

- Using the integrating factor to solve linear differential equations such that $\frac{dy}{dt} + P(t)y = f(t)$.
  - The integrating factor is therefore $e^{\int P(t) dt}$.
  - Multiply both sides by the integrating factor.
  - $e^{\int P(t) dt} \frac{dy}{dt} + e^{\int P(t) dt} P(t)y = e^{\int P(t) dt} f(t)$.
  - then via chain rule ...
  - $\frac{d}{dt} \{ e^{\int P(t) dt} y \} = e^{\int P(t) dt} f(t)$.
  - Then integrate to find solution.

1.9 Integration by Parts

\[ \int udv = uv - \int vdu. \]
2 Systems

- \( \frac{dx}{dt} = ax - bxy, \frac{dy}{dt} = -cy + dxy. \)
- Equilibrium occurs when both differential equations are equal to zero.
- \( a \) and \( c \) are growth effects and \( b \) and \( d \) are interaction effects.
- To verify that \( x(t), y(t) \) is a solution to a system, take the derivative of each and compare them to the original differential equations with \( x \) and \( y \) plugged in.
- Converting a second order differential equation, \( \frac{d^2y}{dt^2} = f. \) Let \( v = \frac{dy}{dt}. \) Thus \( dv = \frac{d^2y}{dt}. \)

2.1 Vector Notation

- A system of the form \( \frac{dx}{dt} = ax + bxy \) and \( \frac{dy}{dt} = cy + exy \) can be written in vector notation.

\[
\frac{d}{dt} \mathbf{P}(t) = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} ax + bxy \\ cy + exy \end{bmatrix}.
\] (1)

2.2 Decoupled System

- Completely decoupled: \( \frac{dx}{dt} = f(x), \frac{dy}{dt} = g(y). \)
- Partially decoupled: \( \frac{dx}{dt} = f(x), \frac{dy}{dt} = g(x, y). \)
3 Systems II

- Matrix form.
- Homogeneous = $\frac{d}{dt}X = AX$.
- Non-homogeneous = $\frac{d}{dt}X = AX + F$.
- Linearity Principal
  - Consider $\frac{d}{dt}X = AX$, where
    \[
    A = \begin{bmatrix}
    a & b \\
    c & d
    \end{bmatrix}. \tag{2}
    \]
  - If $X_1(t)$ and $X_2(t)$ are solutions, then $k_1X_1(t) + k_2X_2(t)$ is also a solution provided $X_1(t)$ and $X_2(t)$ are linearly independent.
  - Theorem: If $A$ is a matrix with $\det A$ not equal to zero, then the only equilibrium point for the system $\frac{d}{dt}X = AX$ is,
    \[
    \begin{bmatrix}
    0 \\
    0
    \end{bmatrix}. \tag{3}
    \]

3.1 Straightline Solutions, Eigencool Eigenvectors and Eigenvalues

- A straightline solution to the system $\frac{d}{dt}X = AX$ exists provided that,
  \[
  A \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}. \tag{4}
  \]
- To determine $\lambda$, compute the $\det[(A - \lambda I)] =
  \[
  \det \begin{bmatrix} a - \lambda & b \\ c & e - \lambda \end{bmatrix} = (a - \lambda)(e - \lambda) - bc = 0. \tag{5}
  \]
  - This expands to the characteristic polynomial =
    \[
    \lambda^2 - (a - d)\lambda + ae - bc = 0.
    \]
- Solving the characteristic polynomial provides us with the eigenvalues of $A$. 
3.2 Stability
Consider a linear 2 dimensional system with two nonzero, real, distinct eigenvalues, \( \lambda_1 \) and \( \lambda_2 \).

- If both eigenvalues are positive then the origin is a source (unstable).
- If both eigenvalues are negative then the origin is a sink (stable).
- If the eigenvalues have different signs, then the origin is a saddle (unstable).

3.3 Complex Eigenvalues

- Euler’s Formula: \( e^{a+ib} = e^a e^{ib} = e^a \cos(b) + ie^a \sin(b) \).
- Given real and complex parts of a solution, the two parts can be treated as separate independent solutions and used in the linearization theorem to determine the general solution.
- Stability: consider a linear two dimensional system with complex eigenvalues \( \lambda_1 = a + ib \) and \( \lambda_2 = a - ib \).
  - If \( a \) is negative then solution spiral towards the origin (spiral sink).
  - If \( a \) is positive then the solutions spiral away from the origin (spiral source).
  - If \( a = 0 \) the solutions are periodic closed paths (neutral centers).

3.4 Repeated Eigenvalues

- Given the system, \( \frac{d}{dt}X = AX \) with one repeated eigenvalue, \( \lambda_1 \).
- If \( V_1 \) is an eigenvector, then \( X_1(t) = e^{\lambda t}V_1 \) is a straight line solution.
- Another solution is of the form \( X_2(t) = e^{\lambda t}(tV_1 + V_2) \).
- Where \( V_1 = (A - \lambda I)V_2 \).
- \( X_1 \) and \( X_2 \) will be independent and the general solution is formed in the usual manner.

3.5 Zero as an Eigenvalue

- If zero is an eigenvector, nothing changes but the form of the general solution is now
  \( X(t) = k_1 V_1 + k_2 e^{\lambda_2 t}V_2 \).
4 Second Order Differential Equations

• Form: \( \frac{d^2y}{dt^2} + p(t) \frac{dy}{dt} = q(t)y = f(t) \).

• Homogeneous if \( f(t) = 0 \).

• given solutions \( y_1 \) and \( y_2 \) to the 2nd order differential equation, you must check the Wronskian if both solutions are from real roots of the characteristic.

\[
W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}.
\] (6)

• If \( W \) is equal to 0 anywhere on the interval of consideration, then \( y_1 \) and \( y_2 \) are not linearly independent.

• General solution given \( y_1 \) and \( y_2 \) is found as usual by the linearization theorem.

• Characteristic polynomial of a 2nd order with constant coefficients: \( as^2 + bs + c = 0 \).

• Solutions of the form \( y(t) = e^{st} \).

\[
s = -\frac{b}{2a} + / - \frac{\sqrt{b^2 - 4ac}}{2a}.
\]

− if \( b^2 - 4ac > 0 \), then two distinct real roots.

− if \( b^2 - 4ac < 0 \), then complex roots.

− \( b^2 - 4ac = 0 \), then repeated real roots.

4.1 Two real distinct Roots

• Two real roots, \( s_1 \) and \( s_2 \).

• General solution = \( y(t) = k_1e^{s_1t} + k_2e^{s_2t} \).

4.2 Complex Roots

• Complex Roots, \( s_1 = p + iq \) and \( s_2 = p - iq \).

• General solution = \( y(t) = k_1e^{pt}\cos(qt) + k_2e^{pt}\sin(Qt) \).

4.3 Repeated Roots

• Repeated Root, \( s_1 \).

• General solution = \( y(t) = k_1e^{-\frac{b}{2a}t} + k_2te^{-\frac{b}{a^2}t} \).
4.4 Nonhomogeneous with constant coefficients

- General solution = \( y(t) = y_h + y_p \).
- Polynomial \( f(t) \).
  - Look for particular solution of the form \( y_p = A t^n + B t^{n-1} + C t^{n-2} + \ldots + D t + E \).
- Exponential \( f(t) \).
  - Look for particular solution of the form \( y_p = A e^{pt} \).
- Sine or Cosine \( f(t) \).
  - Look for particular solution of the form \( y_p = A \sin(at) + B \cos(at) \).
- Combination \( f(t) \).
  - \( f(t) = P_n(t) e^{at} \), \( y_p = (A t^n + B t^{n-1} + C t^{n-2} + \ldots + D t + E) e^{at} \).
  - \( f(t) = P_n(t) \sin(at) \) or \( P_n(t) \cos(at) \), \( y_p = (A_1 t^n + A_2 t^{n-1} + A_3 t^{n-2} + \ldots + A_4 t + A_5) \cos(at) + (B_1 t^n + B_2 t^{n-1} + B_3 t^{n-2} + \ldots + B_4 t + B_5) \sin(at) \).
  - \( f(t) = e^{at} \sin(bt) \) or \( e^{at} \cos(bt) \), \( y_p = A e^{at} \cos(bt) + B e^{at} \sin(bt) \).
  - \( f(t) = P_n(t) e^{at} \sin(bt) \) or \( P_n(t) e^{at} \cos(bt) \), \( y_p = (A_1 t^n + A_2 t^{n-1} + A_3 t^{n-2} + \ldots + A_4 t + A_5) e^{at} \cos(bt) + (B_1 t^n + B_2 t^{n-1} + B_3 t^{n-2} + \ldots + B_4 t + B_5) e^{at} \sin(bt) \).
- Superposition \( f(t) \).
  - If \( f(t) \) is the sum of \( m \) terms of the forms previously described.
  - \( y_p = y_{p1} + y_{p2} + y_{p3} + \ldots + y_{pm} \).
5 LaPlace Transformations

• Definition \( L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \lim_{T \to \infty} \int_0^T e^{-st} f(t) dt. \)

• ONLY PROVIDED THAT THE INTEGRAL CONVERGES!!! MUST BE OF EXPONENTIAL ORDER!!!

• \( L\{f(t)\} = F(s). \)

• \( L\{1\} = \frac{1}{s}. \)

• \( L\{t\} = \frac{1}{s^2}. \)

• \( L\{e^{at}\} = \frac{1}{s-a}. \)

• \( L\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}. \)

• \( L\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}. \)

• Linear: \( L\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s). \)

5.1 Inverse Laplace Transforms

• Linear: \( L^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t). \)

5.2 Transform of a derivative

• \( L\{f'(t)\} = sL(f(t)) - f(0). \)

• \( L\{f''(t)\} = s^2 L(f(t)) - sf(0) - f'(0). \)