Dynamical Systems and Chaos: Mathematics and Economic Applications

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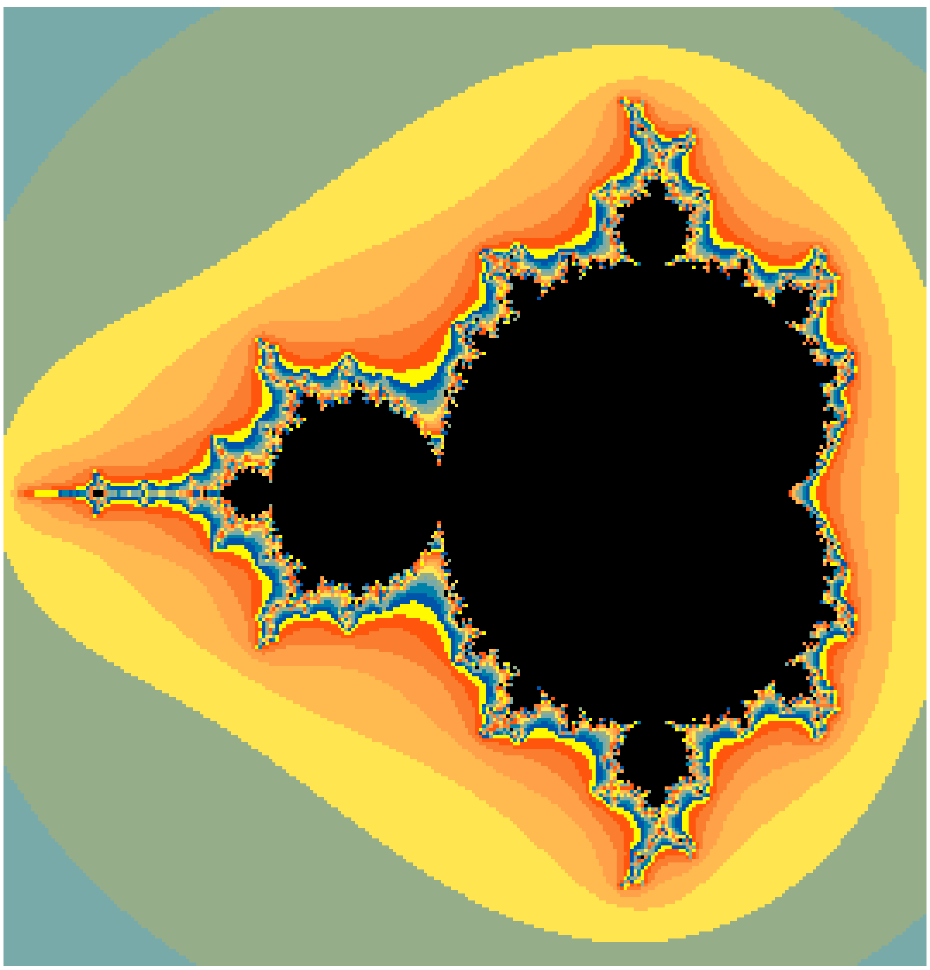

1 Introduction

Dynamical systems provide a way of modeling events that change over time. By understanding the behavior of a system, one can model it with usually a relatively simple set of mathematical equations that can be used as a tool for prediction. The system is dynamic because each outcome depends on one or more of the previous results. Discrete dynamical systems, or difference equations, describe a relationship between one point in time and a previous point. Once the system is determined, one can look at the long-term behavior of the system starting with various initial conditions and fine-tuning the agents in the model. Stability, periodicity, or even chaos can result for different sets of parameters. These types of models have many useful applications and an important one is to economics. In a dynamic economy, economists are constantly trying to predict future behavior of everything from interest rates, to product markets, to consumer behavior.

One of the most interesting long-term behaviors of a dynamical system is when it turns chaotic. Most systems in our everyday life are continuously changing and thus are well modeled by dynamical systems. The discovery of chaos has had a large impact because it reaches into so many different disciplines. Chaos is defined by mathematician and poet, Philip Holmes, as "the complicated, aperiodic, attracting orbits of certain (low dimensional) dynamical systems." (Gleick 306) The details of this definition will be expanded upon later, but it is a start to know that chaos is the complex, non-repeating behavior of usually very simple systems. Edward Lorenz, a meteorologist at MIT, was one of the first to actually see chaos in a machine he created that took several simply discrete dynamical systems and modeled the weather. "Lorenz saw more than randomness embedded in his weather model. His saw a fine geometrical structure, order masquerading as randomness." (Gleick 22)

Before we analyze the intricacies of chaotic systems, we must first cover the basics of dynamical systems and their applications. We will start with simple linear and nonlinear dynamical systems and attempt to clarify the models using economic examples. The following paper analyzes the mathematical structure of dynamical systems and the onset of chaos and then continues on to develop the economic applications of those models.

2 Introduction to Dynamical Systems

The simplest system is a first order discrete dynamical system which is a sequence of numbers $A(n)$ such that each term (after the initial value) is related to the previous term by $A(n+1) = f(A(n))$. The function, f, is some function that relates the current value of the system to the previous value. A good example of this would be an interest rate calculation. Since the balance on an account in the present time period depends on the previous balance plus interest, we can model it as $A(n+1) = (1+r)A(n)$, where r is the rate of interest. Higher order dynamical systems are just those that depend on more than one previous time period. Nonhomogeneous dynamical systems are those in which the coefficients of $A(n)$ depend on some function of n as well. $A(n+1) = f(A(n)) + g(n)$ is a nonhomogeneous system. Let us now consider a simple example of a first order dynamical system, the repayment of a loan. Suppose the initial principal borrowed is $\$10,000$ and the interest on the loan is 5% per year. Suppose initially, the borrower decides to makes payments of \$700 per year (suppose he has this luxury). This can be represented by the following dynamical system, $A(n+1) = (1+0.05)A(n) - 700$. Thus,

$$\begin{aligned} A(0) &= \$10000 \\ A(1) &= 1.05(10,000) - 700 = 9,800 \\ A(2) &= 1.05(9,800) - 700 = 9590 \\ &\cdots \\ A(25) &= 1.05(1,099) - 700 = 455 \\ A(26) &= 1.05(455) - 700 = -223 \end{aligned}$$

And hence after 26 years, the loan would be paid off. Suppose the borrower only chose to pay \$400 per month. Thus,

$$\begin{aligned} A(0) &= \$10,000 \\ A(1) &= 1.05(10,000) - 400 = 10,100 \\ A(2) &= 1.05(10,100) - 400 = 10,205 \\ &\cdots \end{aligned}$$

In this case, the borrower will never pay off the loan as his payments are not exceeding the interest that is accruing on the principal after each year. So the last case we will consider is if he repays \$500 per month.

$$\begin{aligned} A(0) &= \$10,000 \\ A(1) &= 1.05(10000) - 500 = 10000 \\ A(2) &= 1.05(10000) - 500 = 10000 \\ &\cdots \end{aligned}$$

Thus, the payments exactly equal the interest charges and the principal neither increases nor decreases. This value, $a = 10000$, is called an equilibrium value or a fixed point for the dynamical system associated with payments of \$500 since $A(k) = a$ for all k. Thus we have a definition,

Definition 1: A value a is an equilibrium value for a first order dynamical system if and only if $a = f(a)$.

We can generalize the equilibrium value to simple first order systems like $A(n+1) = rA(n) + b$ as $a = \frac{b}{(1-r)}$. If $r = 1$ and $b \neq 0$, then there does not exist an equilibrium value. To see this simply set $a = ra + b$. Thus $a - ra = b$. Thus $a(1-r) = b$. Thus $a = \frac{b}{(1-r)}$.

There are several interesting properties of fixed points or equilibrium values. An equilibrium value is stable or attracting if there is number, ϵ, such that, If $|A(0) - a| < \epsilon$, then $\lim_{k\to\infty} A(k) = a$.

This means that if you have a fixed point a, and you take values within ϵ of it and see what happens, if the systems eventually is "attracted" to the fixed point for all iterations beyond some point, it is stable. Otherwise, if points close to an equilibrium value repel away from it as the system is expanded, then a is unstable. Thus an equilibrium value is unstable or repelling if there is a number ϵ such that if $0 < |A(0) - a| < \epsilon$, then $|A(k) - a| > \epsilon$ for some, but not necessarily all values of k.

Looking back at our simple dynamical system, $A(n+1) = rA(n) + b$, which has an equilibrium value of $a = \frac{b}{(1-r)}$ as long as $r \neq 1$.

Theorem 1: The stability of this point a, if it exists, depends on r such that

- a is stable if $-1 < r < 1$. Thus $\lim_{k\to\infty} A(k) = a$ for all $A(0)$.
- a is unstable if $r < -1$ or $r > 1$. Thus $|A(k)|$ goes to ∞ for all $A(0) \neq a$.
- When $r = 1$, the equilibrium value does not exist as was stated.
- When $r = -1$, the system is a two cycle meaning that $A(0) = A(2) = A(4) = ...$ Since $A(k)$ neither goes to or away from a, a is neutral.

Proof:
This can be shown using induction. We are interested in the distance from the equilibrium point, a, to $A(n)$ evaluated for all n. Consider the distance from $A(1)$ to a. Utilizing the fact that $a = \frac{b}{1-r}$, we find,

$$\begin{aligned}
|A(1) - a| &= |rA(0) + b - \frac{b}{1-r}| \\
&= |rA(0) + \frac{b - rb - b}{1-r}| \\
&= |rA(0) - \frac{rb}{1-r}| \\
&= |r||A(0) - a|.
\end{aligned}$$

Using the same process, we can find the distance from $A(2)$ to a.

$$|A(2) - a| = |r|^2|A(0) - a|.$$

Thus by induction, we can find the general form of the distance from a to $A(k)$.

$$|A(k) - a| = |r|^k|A(0) - a|.$$

Now for $-1 < r < 1$,

$$\lim_{k\to\infty} |r|^k = 0$$

and

$$\lim_{k\to\infty} |A(k) - a| = 0.$$

Thus a is stable as was stated above. If $r > 1$ or $r < -1$,

$$\lim_{k\to\infty} |r|^k = \infty$$

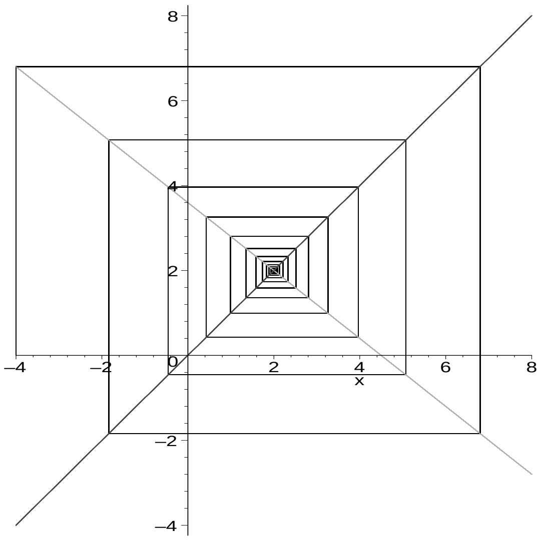

Figure 1: $A(n+1) = -0.8A(n) + 3.6$.

and thus

$$\lim_{k\to\infty} |A(k) - a| = 0.$$

Hence, a is unstable. Finally, suppose $r = -1$. Thus,

$$\begin{aligned} A(n+2) &= -A(n+1) + b \\ &= -(-A(n) + b) + b \\ &= A(n) \end{aligned}$$

The final step was completed by substituting in the formula for $A(n+1)$. Thus we get a 2-cycle when $r = -1$ and this completes the proof. ∎

Graphically, it is relatively easy to determine the stability of an equilibrium point through use of a cobweb graph. A dynamical system is just a function in which the output from the first iteration is plugged back into the function as the input of the second iteration. Graphically if we are given a function of the form, $A(n+1) = f(A(n))$, then first we will graph both $y = f(x)$ and $y = x$ on the same axes. Given a value of $A(0)$, plug that into $f(x)$, which can be done graphically by just extending a line up from $A(0)$ onto $f(x)$. Then from this point, draw a line over to the $y = x$ line. By doing so, we have taken our output from the first iteration and transferred it to a value on the x - axis and it now becomes the input of the function in the second iteration. So continue by drawing a line up to the graph of $f(x)$. Continuing this process a sort of cobweb will be revealed. If by adjusting the initial value $A(0)$, the cobweb always spirals into the equilibrium value, then it is stable. The system $A(n+1) = -0.8A(n) + 3.6$ is shown here in Figure 1 with $A(0) = 4$. It has an equilibrium value of $\frac{b}{(1-r)} = \frac{3.6}{(1--0.8)} = \frac{3.6}{1.8} = 2$.

A 2-cycle system such as $A(n+1) = 3.2A(n) - 2.2A^2(n)$ is shown in Figure 2. The system oscillates between two different values as shown by the two intersections of the cobweb with the

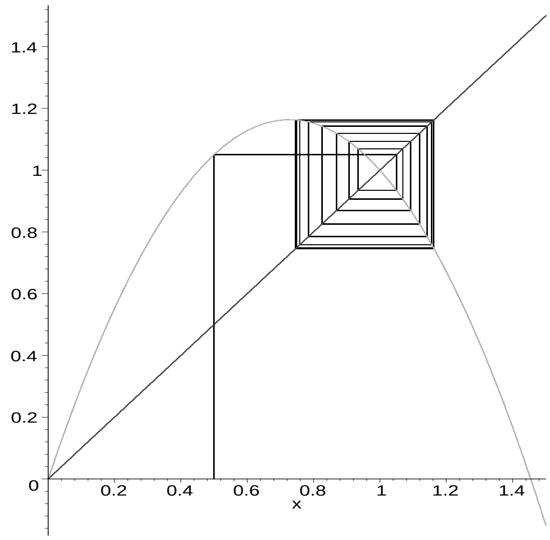

Figure 2: A 2-cycle : $A(n+1) = 3.2A(n) - 2.2A^2(n)$.

function over and over again. The details of a 2-cycle will be explained below.

Now, we will look at a slightly more complex dynamical system, the nonlinear system, $A(n+1) = 1.5A(n) - 0.5A^2(n)$ in Figure 3.

Notice that the point $x = 1$ appears to be stable while the points $x = 0$ appears to be unstable based on the graphical analysis. The key characteristic of the system seems to be the slope of the function at the equilibrium point. This leads directly into a method of determining the stability of a system at a point using derivatives.

Theorem 2: Suppose a is an equilibrium value for the first order dynamical system, $A(n+1) = f(A(n))$. Then a is stable or attracting if $|f'(a)| < 1$ and a is unstable or repelling if $|f'(a)| > 1$.

Proof:

We will now prove the stability of a fixed point, a. First consider an interval around $a, I = (a - \epsilon, a + \epsilon)$. Suppose that $f'(a) < M < 1$ for some $M > 0$. Then for all $x \in I, f'(x) < M < 1$. Now consider $x_0 \in I$ and c in between x_0 and a.

$$|f(x_0) - a| = |f(x_0) - f(a)|,$$

because a is an equilibrium point and thus $f(a) = a$.

$$|f(x_0) - f(a)| = |f'(c)||x_0 - a| \leq M|x_0 - a|.$$

Because the slope of the function at a is less than 1, $f(x^0)$ is closer to a than x_0 and thus $f(x_0) \in I$. Repeating for $f(x_0)$ and using induction, we find,

$$f^n(x_0) - a| \leq M^n|x_0 - a|.$$

Now let $\delta = \epsilon$. If $|x_0 - a| < \delta$, then $|f^n(x_0 - a| \leq M^n|x_0 - a| < \epsilon$. Which proves that a is stable. ▮

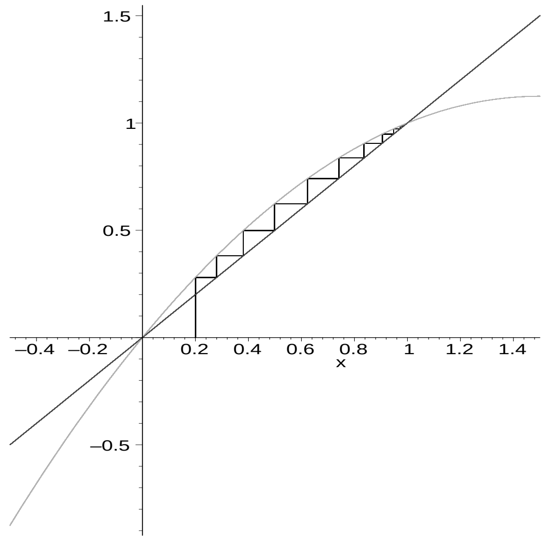

Figure 3: A non-linear dynamical system: $A(n+1) = 1.5A(n) - 0.5A^2(n)$.

This makes intuitive sense as we are bouncing values off the function at the line $y = x$ which obviously has slope $= 1$. Thus if the slope of the function at the equilibrium point, (the best linear approximation to the function), is greater than 1, the point will be repelling, and otherwise, attracting. If $|f'(a)| = 1$, we will eventually look at higher derivatives to determine stability of what is called the "non-hyperbolic case." Before we move on, consider one more dynamical system and examine its equilibrium points. $A(n+1) = 3.2A(n) - 0.8A^2(n)$. To determine the equilibrium points, solve the following equation,

$$\begin{aligned} a &= 3.2a - 0.8a^2. \\ 0 &= 0.8a^2 - 2.2a. \\ a &= 2.75 \text{ or } a = 0. \end{aligned}$$

Graphically this is shown in Figure 4, with an initial value $A(0) = 0.05$ and the first 50 iterations.

It is clear from the graph that initially the equilibrium point of 2.75 appears to be attracting and then it spirals out into a two cycle. The first derivative of $f(x)$ is $f'(x) = 3.2 - 1.6x$. Plugging in 2.75, we find $f'(2.75) = -1.2$ which is less than -1 so 2.75 is unstable. $f'(0) = 3.2$ so 0 is also unstable. However, the derivative test does not tell us anything about the long-term behavior of the system at points other than equilibrium points, only about the stability of those equilibrium points.

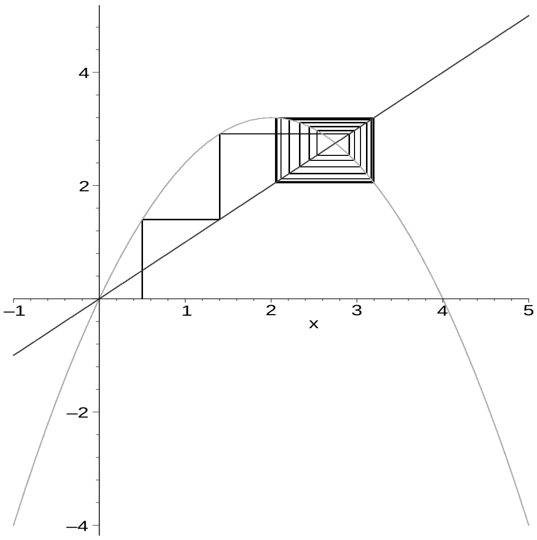

Figure 4: $A(n+1) = 3.2A(n) - 0.8A^2(n)$.

3 First Order Linear Dynamical Systems

We will now consider solutions and applications of dynamical systems. A solution to a dynamical system is a function that satisfies the system for all $k \geq 0$. If every solution to a dynamical system is of the form of a function, then this function is a general solution. For example in our interest rate example earlier, $A(n+1) = (1+r)A(n)$. The general solution is $A(k) = c(1+r)^k$. A particular solution, or example, would then be for $A(0) = 4$, $4 = c(1+r)^0$, then $c = 4$. Thus the particular solution is $A(k) = 4(1+r)^k$. A solution of $A(k)$ is periodic if $A(k+m) = A(k)$ for some m and k. m is therefore the period of the solution. The general solution to the following system,

$$A(n+1) = rA(n) + b$$

is $A(k) = cr^k + a$ if r does not equal 1 and thus the particular solution to the system is

$$A(k) = (r)^k(a(0) + \frac{b}{(1-r)}) - \frac{b}{(1-r)}.$$

There are many applications of dynamical systems to economics and finance such as interest rate calculations which was discussed before, present value and future value of money, and also calculations involving an annuity. Solving the last equation for $a(0)$, we have a formula for calculating the principal needed to satisfy a certain type of annuity. Solving for $a(0)$,

$$a(0) = (A(k) + b/r)(1+r)^{-k} - \frac{b}{r}.$$

Consider the following example. Suppose one is looking to retire and estimates that they will need \$40,000 per year to retire on. Suppose also that the interest rate at the local bank is 10% and this person expects to live 25 years after retirement. How much does this person need to have in the bank at the time of retirement so that after 25 years, his savings are depleated? ($A(25) = 0$) The solution can be found using a first order dynamical system,

$$A(n+1) = (1+0.1)A(n) - 40,000.$$

The solution to this system can be found using the method above yielding,

$$A(k) = (1+0.1)^k a(0)$$

. Thus,

$$a(0) = -400,000(1.1)^{-25} + 400,000 = \$363,081.$$

$400,000 is used because that is the amount of money would needed to be able to withdraw $40,000 per year for eternity at 10% interest. Thus we conclude that the person must have saved $363,081 to ensure an income level of $40,000 per year for 25 years and end up with $0 at the end of his life.

3.1 Supply and Demand Analysis

One of the more interesting, and in my opinion, the coolest, applications of dynamical systems involves supply and demand analysis in economics. All market are made up of suppliers and demanders, producers and consumers. We can model their behavior as well as price using dynamical systems and then determine the long-term behavior of the market. Consider the following set of equations.

- $S(n+1) = 0.8P(n)$
- $D(n+1) = -1.2P(n+1) + 20$
- $S(n+1) = D(n+1)$

The first equation relates supply in the next time period to price in the current time period. They are positively related as if there is a high price in one year, supplier will be eager to produce a large amount in the next year. The second equation relates demand and price and has a negative slope. This demonstrates the law of demand: as price rises, quantity demanded falls. The coefficients 0.8, -1.2 and 20 are arbitrary to the problem though their sign and magnitude are significant. For example supply next year is positively related to price in the current year. If $S(n+1)$ and $P(n)$ were both in natural logs, then the coefficient of $P(n)$ would be the price elasticity of supply. Finally there is the equilibrium condition, which states that demand in the next period must equal supply. Market forces guarantee that the price in a period will adjust so that supply equals demand. Substituting the first two equations into the third,

$$\begin{aligned} S(n+1) &= D(n+1) \\ 0.8P(n) &= -1.2P(n+1) + 20 \\ P(n+1) &= -(2/3)P(n) + (\frac{50}{3}) \end{aligned}$$

This last equation is a dynamical system that models the fluctuations in price in this system. We can determine the equilibrium value as follows:

$$\begin{aligned} P &= -(\frac{2}{3})P + \frac{50}{3} \\ P &= 10. \end{aligned}$$

Since the r value in the above equation is $-\frac{2}{3}$, we find that the general solution to this system is

$$P(k) = (-2/3)^k c + 10.$$

Since $|r| < 1$, the equilibrium value is stable and though the price in this market might fluctuate around $p = 10$, it will eventually settle down to this price over a long period of time. Looking at a more general case, we realize something important about producer and consumer behavior and the resulting influence on the market price. Consider the following three general equations.

- $S(n+1) = sP(n) + a$
- $D(n+1) = -dP(n+1) + b$
- $S(n+1) = D(n+1)$

These are just the general forms of the equations studied above and have the same characteristics. Small s in the first equation is called the sensitivity of producers to price. Small d in the second equation is the sensitivity of the consumers to price. Setting the equations equal to each other as before,

$$\begin{aligned} S(n+1) &= D(n+1) \\ sP(n) + a &= -dP(n+1) + b \\ P(n+1) &= -(\frac{s}{d})P(n) + \frac{b-a}{d}. \end{aligned}$$

The equilibrium value can also be found as before,

$$\begin{aligned} p &= -(\frac{s}{d})p + \frac{b-a}{d} \\ p + (\frac{s}{d})p &= \frac{b-a}{d} \\ p(1 + \frac{s}{d}) &= \frac{b-a}{d} \\ p &= \frac{\frac{b-a}{d}}{1 + \frac{s}{d}} \\ p &= \frac{b-a}{d+s} \end{aligned}$$

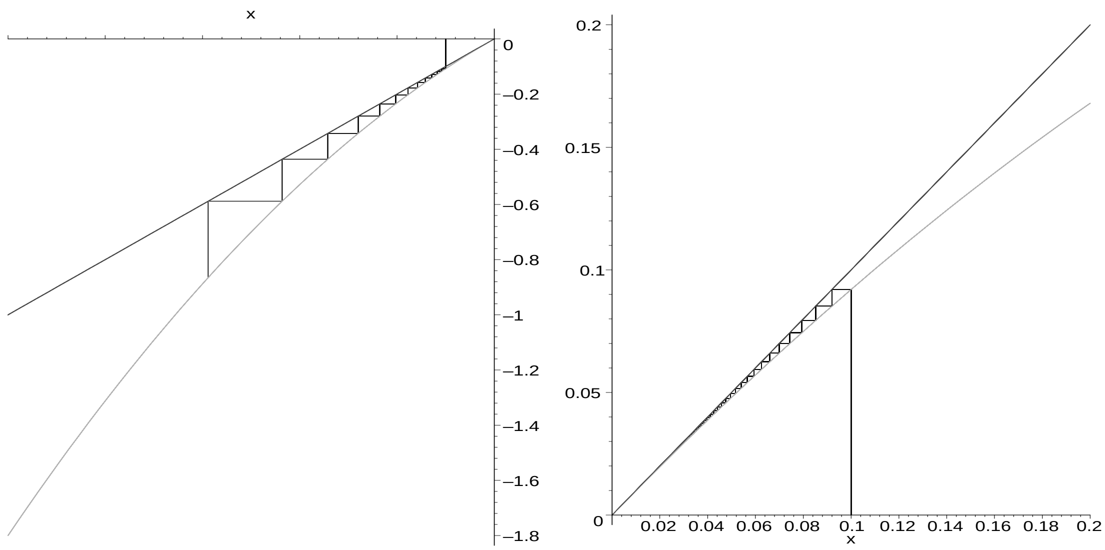

Figure 5: A semistable fixed point

Thus the general solution to this system is,

$$P(k) = c(-\frac{s}{d})^k + \frac{b-a}{d+s}.$$

Economists are not necessarily interested in the equilibrium value, possibly because it is very difficult to determine. They are more concerned with the stability of that value. We know from previous examples, that to determine stability one looks at the r value or in this case $-\frac{s}{d}$. Since s and d are both positive, $-\frac{s}{d} < 0$. Thus we know that the equilibrium value is stable if $-\frac{s}{d} > -1$ or $s < d$ (Consumers more sensitive). The value is unstable if $-\frac{s}{d} < -1$ or $s > d$ (Producers more sensitive). Thus we have shown the the sensitivity of price is the defining factor in determining if the price in a market is stable. Formally this result is called the Cobweb Theory of Economics. Small s and d are difficult to approximate but their relationship is sometimes known and this information allows firms as well as governments to better predict market behavior and take necessary steps to avoid large price fluctuations.

4 Nonlinear Dynamical Systems - Complex Behavior

We will now return to the non-hyperbolic case when $|f'(a)| = 1$. This provided us with inconclusive evidence of the stability of a system. Consider the following dynamical system.

$$A(n+1) = A(n) - 0.8A^2(n).$$

The system is plotted in Figure 5.

As shown by these two graphs, since 0 is an equilibrium point for the system, the cobweb graph demonstrates the stability of those points. At an initial value to the left of $0, A(0) = -0.05$, the equilibrium point is repelling as shown by the left graph. At a point to the right of $0, A(0) = 0.05$, the equilibrium point is attracting as shown by the right hand graph. Therefore an equilibrium point is called semistable as it differs depending on which side the initial value is on. Notice the concavity of the function at the point is the defining characteristic of this type of stability. This leads us to the following theorem.

Theorem 3: Suppose that a is an equilibrium point for the dynamical system $A(n+1) = f(A(n))$, and that $f'(a) = 1$, then

- If $f''(a)$ does not equal 0, a is semistable.
- If $f''(a) = 0$, then:
  - If $f'''(a) < 0$, the equilibrium point is stable.
  - If $f'''(a) > 0$, the equilibrium point is unstable.

Proof:
We will now show that a is unstable if $f'(a) = 1$ and $f''(a) \neq 0$. Thus, $f''(a)$ is either greater than 0 or less than 0.

- Assume $f''(a) > 0$. Thus $f'(a)$ is increasing over the interval $I = (a, a+\delta)$ for $\delta > 0$. Thus $f'(a) > 1$ over I. By Theorem 2 above, a is unstable.
- Assume $f''(x^*) < 0$. Thus $f'(a)$ is decreasing over the interval $I = (a-\delta, a)$ for $\delta > 0$. Thus $f'(a) > 1$ over I. Again Theorem 2 above, a is unstable. This completes the proof.

∎

This is because if $f''(a) = 0$, there is a point of inflection and thus the concavity changes on either side of the point. Thus we must look to the third derivative to determine the stability of the point. Consider the following example,

$$A(n+1) = A^4(n) - 2A^3(n) + 3A(n) - 1.$$

Solving to find the equilibrium points,

$$\begin{aligned}
a^4 - 2a^3 + 3a - 1 &= a \\
a^4 - 2a^3 + 2a - 1 &= 0 \\
(a-1)(a^3 - a^2 - a + 1) &= 0 \\
(a-1)(a-1)(a^2 - 1) &= 0 \\
(a-1)(a-1)(a-1)(a+1) &= 0 \\
(a-1)^3(a+1) &= 0 \\
a = 1 \text{ and } a = -1
\end{aligned}$$

So this system has equilibrium points at $+1$ and -1. Now we can use the following function and its deriviatives to determine stability.

$f(x) = x^4 - 2x^3 + 3x - 1$	$x = -1$	$x = 1$
$f'(x) = 4x^3 - 6x^2 + 3$	$f' = -7$	$f' = 1$
$f''(x) = 12x^2 - 12x$	$f'' = 24$	$f'' = 0.$
$f'''(x) = 24x - 12$	$f''' = -36$	$f''' = 12.$

As shown in the table, at $x = -1$, we need only look at the first derivative and since it less than -1, the equilibrium point is repelling. At $x = 1$, we must appeal to theorem 3 since the first derivative is 1 and the second is 0. Since $f'''(1) = 12 > 0, x = 1$ is unstable. The method of determining the stability at an equilibrium point if $f'(a) = -1$ is slightly more complicated but the result is shown here.

Theorem 4: Suppose that a is an equilibrium point for the dynamical system $A(n+1) = f(A(n))$, and that $f'(a) = -1$, then

- If $-2f'''(a) - 3[f''(a)]^2 < 0$, then a is stable.
- If $-2f'''(a) - 3[f''(a)]^2 > 0$, then a is unstable.

Proof:
Assume a in an equilibrium point for the system and $f'(a) = -1$. Create a function g, such that, $g = f \circ f = f^2$. Note that since a is a fixed point for f, it is also a fixed point for g. Thus,

$$g'(x) = \frac{d}{dx} f(f(x)) = f'(f(x))f'(x).$$

Hence, since $f(a) = a$ and $f'(a) = -1$,

$$g'(a) = [f'(a)]^2 = 1.$$

Now compute, $g''(a)$.

$$g''(x) = f'(f(x))f''(x) + f''(f(x))[f'(x)]^2.$$

Thus,

$$g''(a) = f'(a)f''(a) + f''(a)[f'(a)]^2.$$

Since $f'(a) = -1$,

$$g''(a) = -1(f''(a)) + 1(f''(a)) = 0.$$

Thus,

$$g'''(a) = -2f'''(a) - 3[f''(a)]^2.$$

From Theorem 3 above, it now follows that if $g'''(a) > 0$, a is stable, and otherwise, a is unstable. ∎

So in the previous sections we have done a detailed analysis of ways in which to determine the stability of an equilibrium point in a system. If the point is stable, we know exactly what will happen to the system given initial values "close" to that value. If the system is unstable, the system might go off to infinity, but it also might become periodic. We have discussed a type of behavior called a 2-cycle but now we will define it more formally.

Definition 2: Two numbers form a 2-cycle for a dynamical system if $A(n) = a_1$, $A(n+1) = a_2$, and $A(n+2) = a_1$. A 2-cycle is stable if there exists intervals (c_1, d_1) and (c_2, d_2) such that,

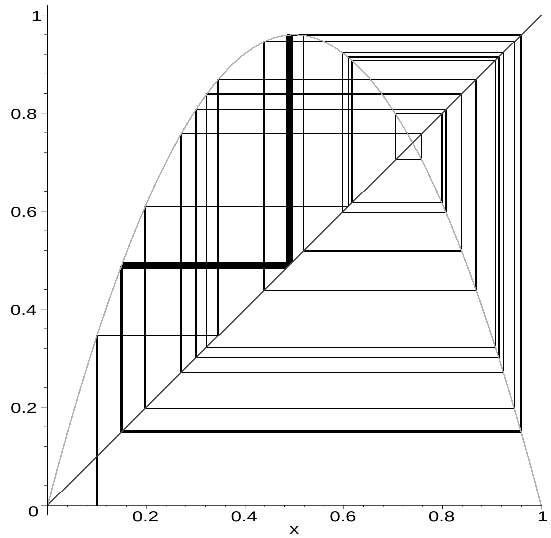

Figure 6: A 3-cycle: $A(n+1) = 3.84A(n) - 3.84A^2(n)$.

- If $A(0)$ is in (c_1, d_1), then $\lim_{k->\infty} A(2k) = a_1$ and $\lim_{k\to\infty} A(2k+1) = a_2$.
- If $A(0)$ is in (c_2, d_2), then $\lim_{k->\infty} A(2k) = a_2$ and $\lim_{k\to\infty} A(2k+1) = a_1$.

This just means that if $A(0)$ is close to the range of the cycle then the system will be attracted to the cycle with its corresponding even or odd terms.

To determine if a system is a 2-cycle algebraically, calculate $f(f(a))$ and if it equals a then either a is an equilibrium value or the system is a 2-cycle with a as one of its values. This analysis can also be extended to larger cycles and generalized for a k-cycle. A further method of determining the stability of a k-cycle in a dynamical system is as follows:

Theorem 5: Suppose a dynamical system has a k-cycle consisting of the numbers $a_1, a_2, ..., a_k$. The k cycle is

- stable if $|f'(a_1)f'(a_2)...f'(a_k)| < 1$
- unstable if $|f'(a_1)f'(a_2)...f'(a_k)| > 1$.

This is fine for determining stability of the cycles but it is not often the case that one knows the values of the cycle or even if the cycle exists. The following is an example of a 3-cycle with $A(0) = 0.1$ shown in Figure 6.

$$A(n+1) = 3.84A(n) - 3.84A^2(n)$$

.

The darker black line begins to outline the 3-cycle. Running the system out further, we obtain approximations to the 3 cycle,

$$\begin{aligned}
x[140] &= 0.4879776401 \\
x[141] &= 0.9594449776 \\
x[142] &= 0.149415600 \\
x[143] &= 0.4880278214 \\
x[144] &= 0.9594496007 \\
x[145] &= 0.149399288
\end{aligned}$$

We also know that $f'(x) = 3.84 - 7.69x$. Plugging in these approximate values into the first derivative and multiplying, we find that

$$|f'(0.49) * f'(0.96) * f'(0.15)| = 0.02.$$

Since $0.02 < 1$, by the previous definition, this 3-cycle is stable.

Finally, we will look at the logistic equation, $A(n+1) = rA(n)(1 - A(n))$. This equation is often used to model population growth rates in biology. Depending on the value of r, this system has many interesting properties. As r is increases, the systems goes through bifurcations as the period of its cycle doubles.

- For $1 < r < 3, A(n)$ has one attracting equilibrium value.
- For $3 < r < 3.45, A(n)$ has an attracting 2-cycle.
- For $3.45 < r < 3.65, A(n)$ has an attracting 4-cycle.
- And so on...

These bifurcations will continue as r is increases. Small r represents a sort of quantitative measure of the amount of nonlinearity in the model. Feigenbaum noticed, "the period doublings were not just coming faster an faster, but they were coming faster and faster at a constant rate." (Gleick 172) He calculated the ratio of convergence to be 4.6692016090, referred to as the Feigenbaum constant. Since the bifurcations are occurring at an accelerating rate, the values for r at which they occur must be bounded. In fact, r never has to be any larger than 3.57 to capture an infinite number of period doublings. This ratio brings in the idea of scaling and is clearly ties in with the self-similarity on all scales when we look at fractal images which is mentioned shortly. The ratio applies to all dynamical systems that go through period doublings on the way to chaos. The following are a few graphs of the period doublings that occur in the logistic equation.

The blue diamonds in Figure 7 demonstrate the single attracting equilibrium points when $r = 2.5$. The pink squares display the 2-cycle as $r = 3.2$ and the yellow triangles show a 4-cycle when $r - 3.5$.

The graph in Figure 8 shows the apparent randomness of the logistic equation as r is increased to 3.57. Beyond this limit, apparent disorder seems to occur, but there are still peculiarities that

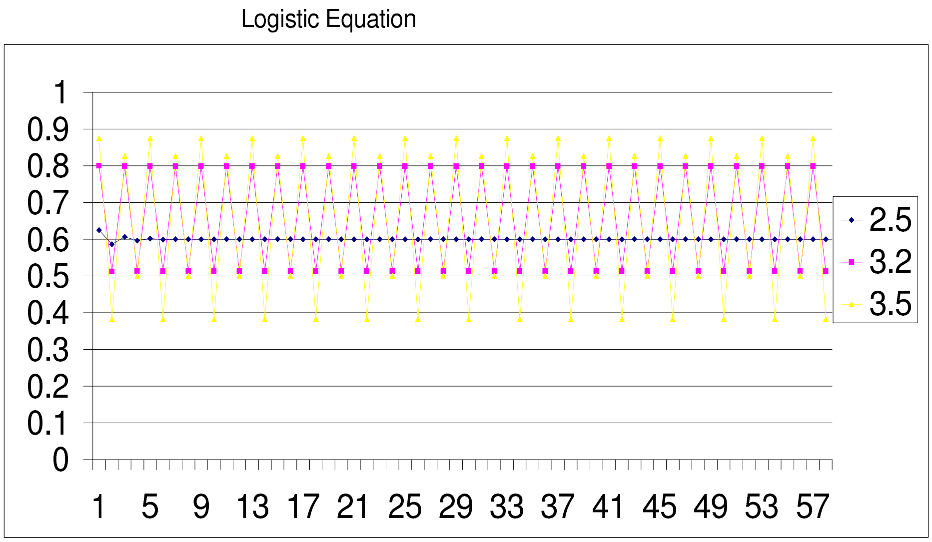

Figure 7: Logistic Equation for $r = 2.5$, 3.2, and 3.5

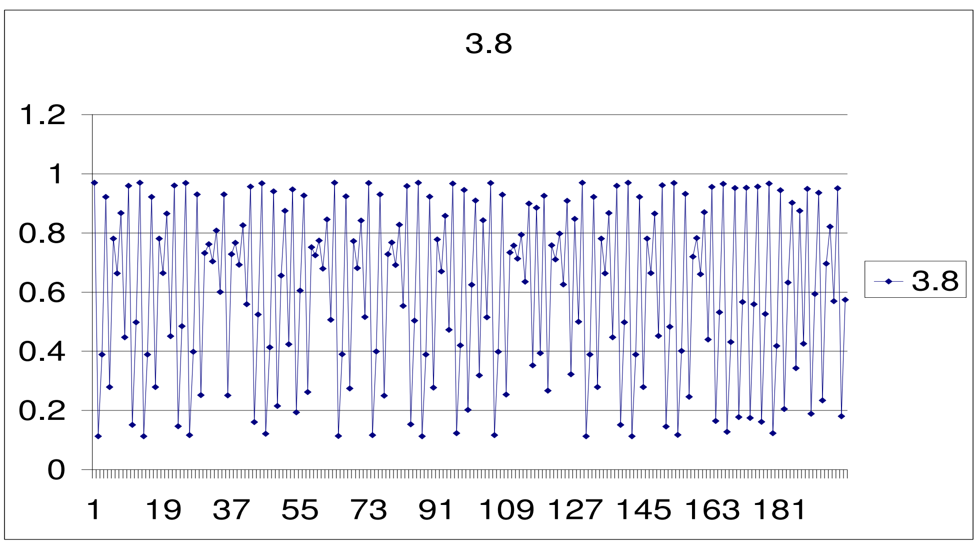

Figure 8: Logistic equation for $r = 3.8$

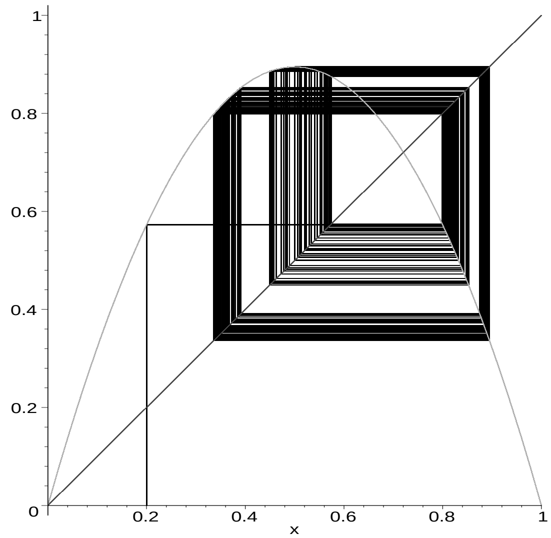

Figure 9: 1st order chaotic system: $A(n+1) = 3.58A(n)(1 - A(n))$.

are seen such as the period doublings starting over again, which seems to imply some sort of self-similarity that is appearing.

The following, in Figure 9, is a cobweb which displays another interesting characteristic of these complex systems.

$$A(n+1) = 3.58A(n)(1 - A(n)) \text{ with } A(0) = 0.2.$$

Such a chaotic system displays sensitive dependence on initial conditions: Whenever you take two initial values a_0 and b_0, which are close together, then $A(k)$ and $B(k)$ eventually get further apart. More rigorously, there exists ϵ, such that, if $0 < |a_0 - b_0| < \epsilon$, then there exists k, such that, $|A(k) - B(k)| > \epsilon$.

The plot, Figure 10, demonstrates the bifurcations that occur to the logistic equation as r is increased from 2.5 to 4. These rapid bifurcations bring to mind another idea in chaos theory called an attractor. An attractor is simply a set of points that a system is drawn to as time passes. For instance 0 is an attractor for the system, $y_{t+1} = 0.5y_t$. A 2-cycle can also be an attractor as points in the neighborhood of those points are drawn into the cycle. For chaotic systems, the resulting attractor is called a "strange attractor." A strange attractor is special, and strange, in that it is an uncountable set of points, such that all points on or around the attractor are pulled in and the resulting time path is aperiodic, but stable. This would imply that it never repeats itself but also never leaves a finite space. It would have to be a fractal. Lorenz's weather experiment involved 3 equations and thus the resulting attractor could be displayed in 3-space. Figure 11 displays what it looked like, the first ever strange attractor known as the "Lorenz Attractor" or simply, "butterfly wings."

In Economics, there is a clear case for using dynamical systems to model economic variables and behavior. Samuelson's multiplier-accelerator model is a perfect example. He believed that current income (Y_t)is split up between consumption (C_t) and investment (I_t). Current consumption is a

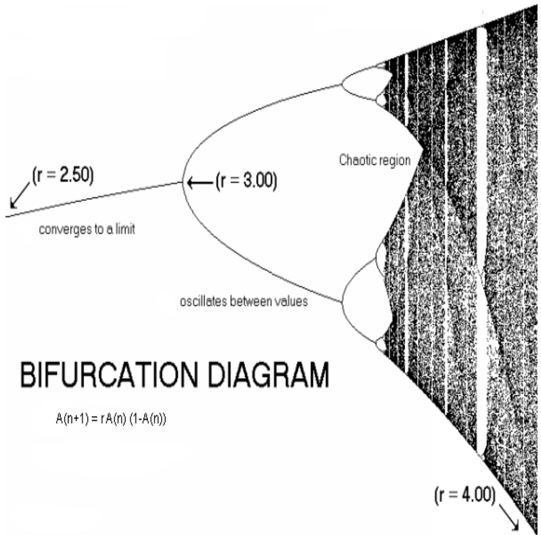

Figure 10: Bifurcations on the way to chaos

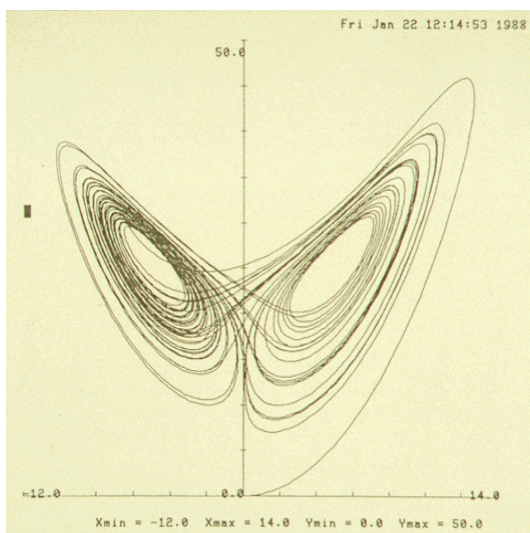

Figure 11: The Lorenz Attractor

function of income in the previous period multiplied by the marginal propensity to consume, c. Investment is a function of the change in income over the past two periods.

$$\begin{aligned} Y_t &= C_t + I_t \\ C_t &= cY_{t-1} + k \\ I_t &= b(Y_{t-1} - Y_{t-2}) \end{aligned}$$

Thus substituting in,

$$Y_t = (c + b)Y_{t-1} - bY_{t-2} + k.$$

Clearly Y_t is a second order linear dynamical system. Varying the coefficients only slighty can produce large variations in the long term behavior of the model and thus the prediction value of such a model is poor. Though the assumptions do seem accurate and the resulting relationship, Y_t, appears to be reasonable, there are still problems. As econometricians attempt to fit regression coefficients to b, c, and k, they are usually not robust and often insignificant. So, why do economists, (and other researchers such as biologists and meterologists) try to use these dynamical systems to model behavior. Specifically, why do non-linear models often result in chaos? The answer can be shown by the following simple example. Consider a firm whose profits are a function of its advertising budget. Assume that if \$0 is spent on advertising, the firm will have \$0 profits. Then profits will increase with advertising until some level and then eventually experience decreasing returns beyond some point. The resulting system would be appropriate.

$$P_t = ay_t(1 - y_t).$$

P_t is the total profit and y_t is the fraction of total expenditure for the firm that goes to advertising at time t. The resulting curve is a hill shaped curve like we have been studying. Now consider that expenditure on advertising at time t is related to profits in the previous time period.

$$y_{t+1} = bP_t.$$

Thus,

$$\begin{aligned} \frac{y_{t+1}}{b} &= ay_t(1 - y_t). \\ y_{t+1} &= aby_t(1 - y_t). \end{aligned}$$

The resulting model is of the chaotic form that experiences sensitive dependence on initial conditions and thus makes forecasting very difficult. Again, two very simple assumptions and the resulting equations, has the possiblity of exhibiting chaotic characteristics. Chaos theory "demonstrates dramatically the dangers of extrapolation and the difficulties that can beset economic forecasting generally." (Baumol 80). However, given how easily and frequently this type of model shows up, further study into nonlinar dynamic models is necessary.

Chaotic systems have the characteristic of exhibiting many unstable periodic cycles. Scientists have always looked for and assumed stable systems because any system that is unstable is too quickly brought to a point of stability at the slightest disturbance. But chaos changed all that. "A chaotic system could be stable if its particular brand of irregularity persisted in the face of small disturbances ... the system would return to the same peculiar pattern or irregularity as before. It was locally unpredictable, globally stable." (Gleick 48) It may seem that once the amount of nonlinearity in the model, the value of r, is increased to a point in which the system exhibits chaos, that we have lost all value and the model is useless. Since systems like the logistic growth equation are often used to model actual populations and other real world situations, the discovery of chaos shows us that although these systems are not predictable, because of the period doublings and attractors inherent in the models, they do show the cyclical and self-similar characteristics of just about every system in the world around us.

The beauty of chaos is best described by the following quote by Gert Eilenberger, a German physicist. "Why is it that the silhouette of a storm-bent leafless tree against an evening sky in winter is perceived as beautiful, but the corresponding silhouette of any multi-purpose university building is not, in spite of all efforts of the architect? The answer seems to me, even if somewhat speculative, to follow from the new insights into dynamical systems. Our feeling for beauty is inspired by the harmonious arrangement of order and disorder as it occurs in natural objects in clouds, trees, mountain ranges, or snow crystals. The shapes of all these are dynamical processes jelled into physical forms, and particular combinations of order and disorder are typical for them." (Gleick 117)

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